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Acyclic List Edge Coloring of Graphs

Ko-Wei Lih

Institute of Mathematics
Academia Sinica

A Joint Work with Hsin-Hao Lai

Overview

- Acyclic edge coloring
- Acyclic list edge coloring
- Useful lemmas
- Outerplanar graphs
- Subcubic graphs
- Attaching a cycle
- Halin graphs
- Open problems

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- Acyclic edge coloring
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All graphs in this talk are finite, without loops or parallel edges.

The chromatic number $\chi(G)$ of G is the least number of colors needed in a proper vertex coloring of G.

The chromatic index $\chi'(G)$ of G is the least number of colors needed in a proper edge coloring of G.

A proper coloring of the vertices (or edges) of a graph G is called acyclic if there is no 2-colored cycle in G.

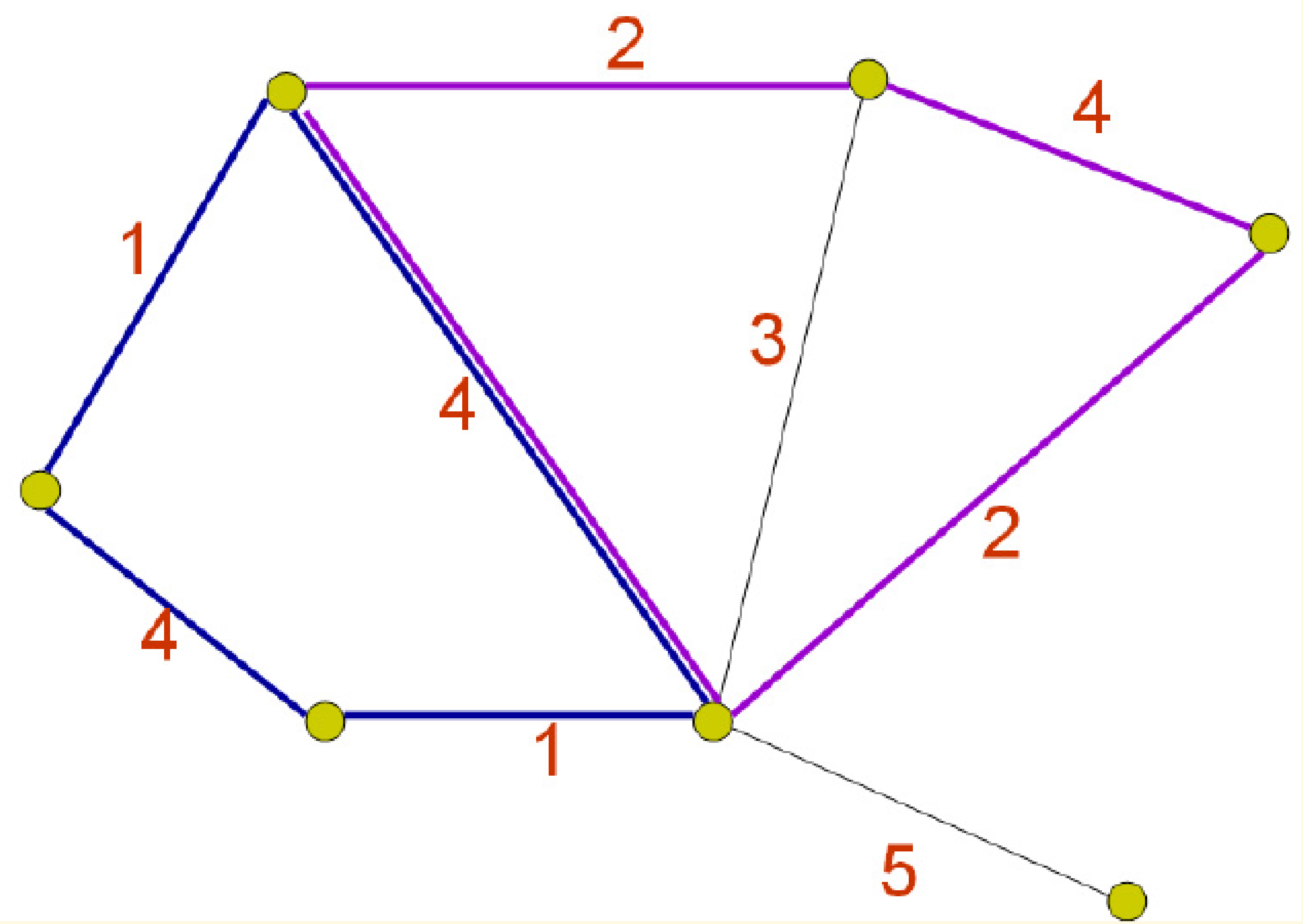
$\Leftrightarrow$

Every cycle of G is colored with at least 3 colors.

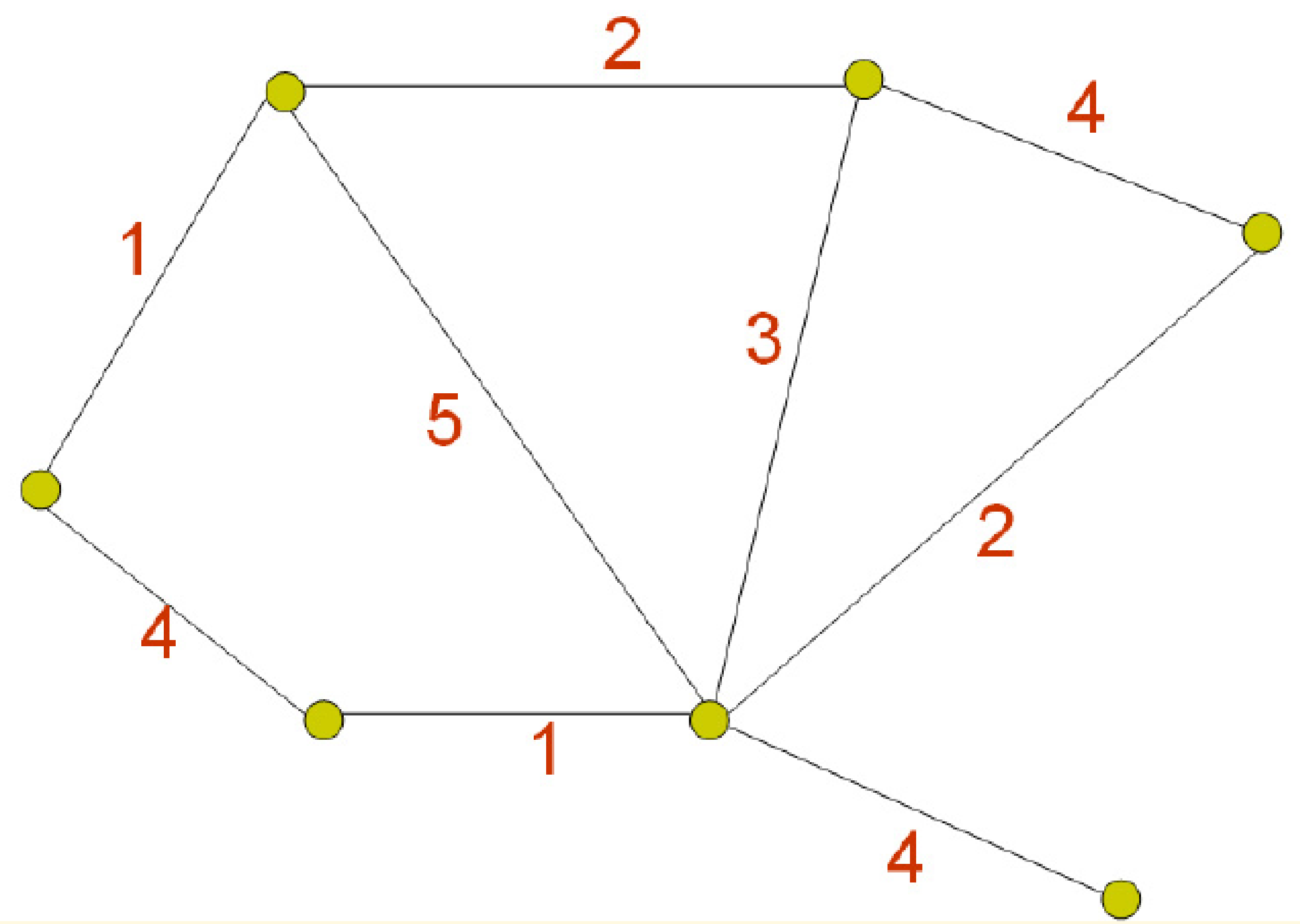
$\Leftrightarrow$

The union of any two color classes induces a subgraph of G that is a forest.

5-edge coloring



acyclic 5-edge coloring



The acyclic chromatic number $a(G)$ of G is the least number of colors in an acyclic vertex coloring of G.

The acyclic chromatic index $a'(G)$ of G is the least number of colors in an acyclic edge coloring of G.

Vizing's Theorem (1964)

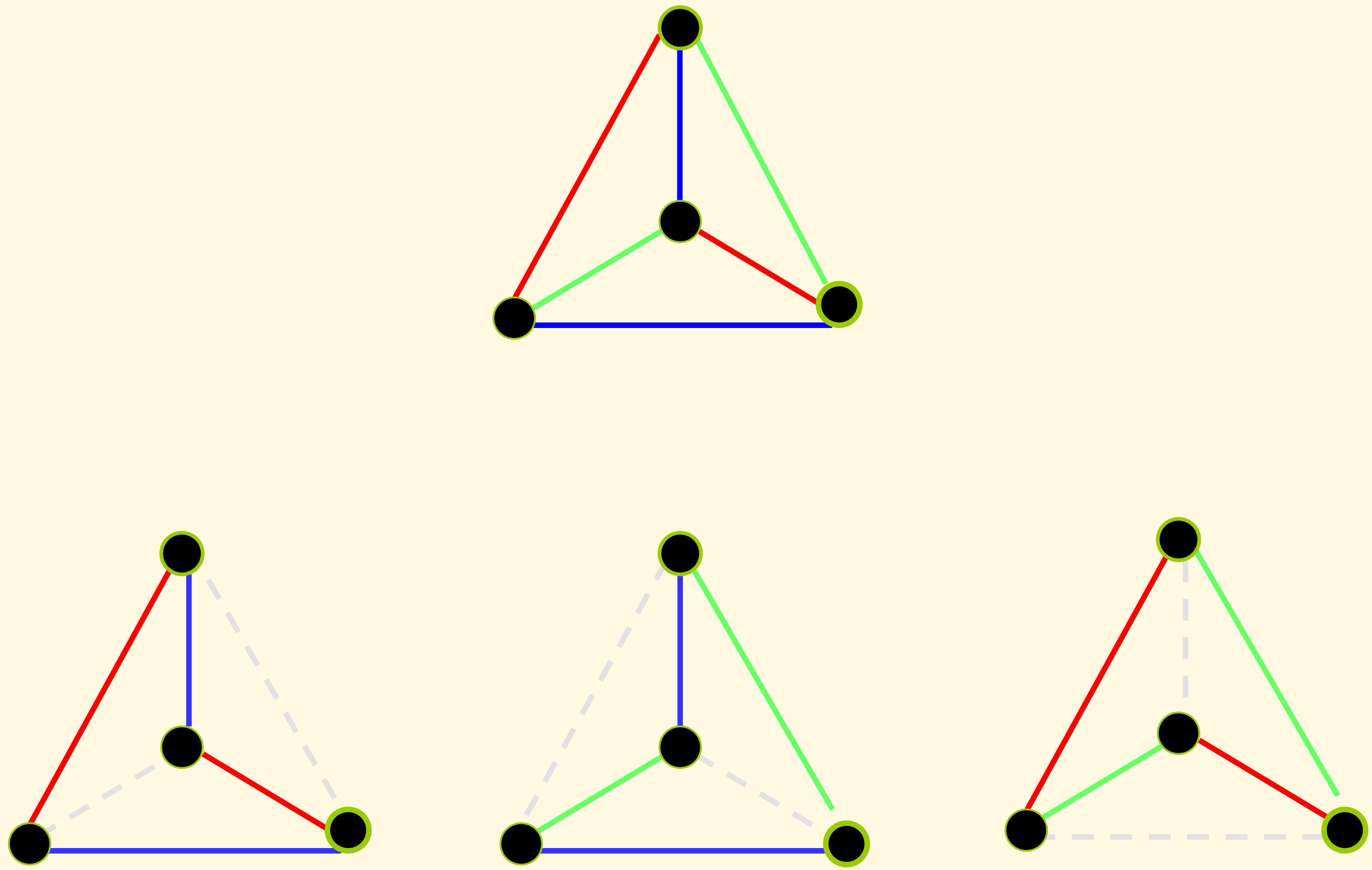
$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

$\Delta(G)$: the maximum degree of G

Question: $a'(G) \leq \Delta(G) + 1$?

No!

$a'(K_{2n}) > \Delta(K_{2n}) + 1 = 2n$ for $n \geq 2$.



General upper bounds for $a'(G)$

Molloy, Reed (1998):

$a'(G) \leq 16\Delta(G)$

Muthu, Narayanan, Subramanian (2005): When the girth, the length of a shortest cycle, of G is at least 220,

$a'(G) \leq 4.52\Delta(G)$

Acyclic Edge Coloring Conjecture:

$$a'(G) \leq \Delta(G) + 2$$

Proposed independently by

Fiamčík in 1978 and

Alon, Sudakov, Zaks in 2001.

Fiamčík (1984): If $\Delta(G) \leq 3$ and G has neither K_4 nor $K_{3,3}$ as a component, then $a'(G) \leq 4$, whereas $a'(K_4) = a'(K_{3,3}) = 5$.

Alon, Sudakov, Zaks (2001): There exists a constant c such that $a'(G) \leq \Delta(G) + 2$ for any G whose girth is at least $c\Delta(G)\log\Delta(G)$.

Alon, Zaks (2002): It is NP-complete to determine whether $a'(G) \leq 3$.

Skulrattankulchai (2004): A polynomial time algorithm to color a subcubic graph using 5 colors.

Nešetříl, Wormald (2005):

$a'(G) \leq \Delta(G) + 1$ for a random Δ–regular graph.

Muthu, Narayanan, Subramaniann (2005):

$a'(G) \leq \Delta(G) + 1$ if G is a partial 2-tree, an outerplanar graph, or a partial torus.

Basavaraju, Sunil Chandran (2008):
$a'(G) \leq \Delta(G) + 1$ if G is a 2-degenerate graph.

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An edge-list L assigns a finite set of positive integers to each edge of G.

Let f: $E(G) \rightarrow \mathbf{N}$. An edge-list L is an f-edge-list if $|L(e)| \geq f(e)$ for every edge e.

An acyclic edge coloring φ of G satisfying $\varphi(e) \in L(e)$ for every edge e is called an acyclic L-edge coloring of G.

A graph G is said to be acyclically f-edge choosable if it has an acyclic L-edge coloring for any f-edge-list L.

The acyclic list chromatic index $a_{list}'(G)$ is the least integer k such that G is acyclically k-edge choosable.

$$\Delta(G) \leq \chi'(G) \leq a'(G) \leq a_{list}'(G).$$

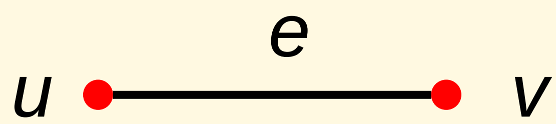
Lemma. Assume that f_1 and f_2 are two mappings from $E(G)$ to $\mathbf{N}$ such that $f_1(e) \leq f_2(\mathrm{e})$ for each e. If G is acyclically f_1-edge choosable, then G is acyclically f_2-edge choosable.

Lemma. If H is a subgraph of a graph G, then $a_{\mathrm{list}}'(H) \leq a_{\mathrm{list}}'(G)$.

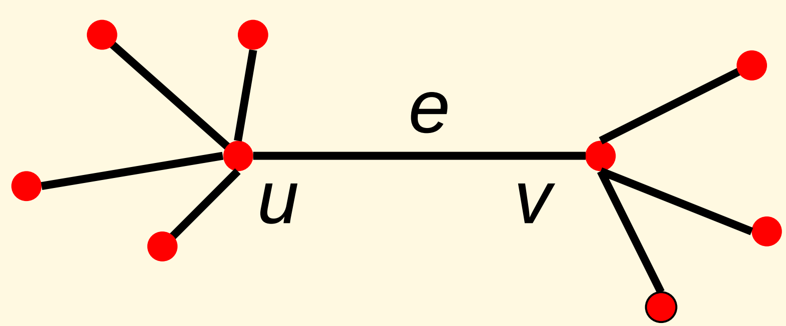
Lemma. If $G_1, G_2, \ldots, G_k$ are all the components of G, then $a_{\mathrm{list}}'(G) = \max\{a_{\mathrm{list}}'(G_1), a_{\mathrm{list}}'(G_2), \ldots, a_{\mathrm{list}}'(G_k)\}$.

Let $e = uv$ be an edge of the graph G. Let $N_0(e)$ and $N_1(e)$ denote the sets $\{u, v\}$ and $\{x : xu \in E(G) \text{ or } xv \in E(G)\}$, respectively.

$N_0(e)$



$N_1(e)$



Let $e = uv$ be an edge of the graph G. Let $N_0(e)$ and $N_1(e)$ denote the sets $\{u, v\}$ and $\{x : xu \in E(G)$ or $xv \in E(G)\}$, respectively.

For $i = 0$ and 1, let Δ_i denote the mapping

$$\Delta_i(e) = \max\{\deg(x) : x \in N_i(e)\}$$

for each edge e.

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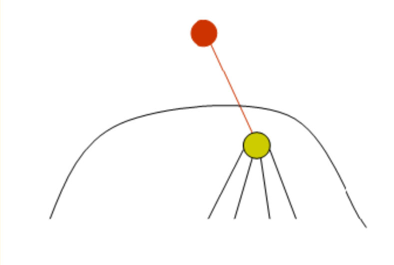
Adding a leaf

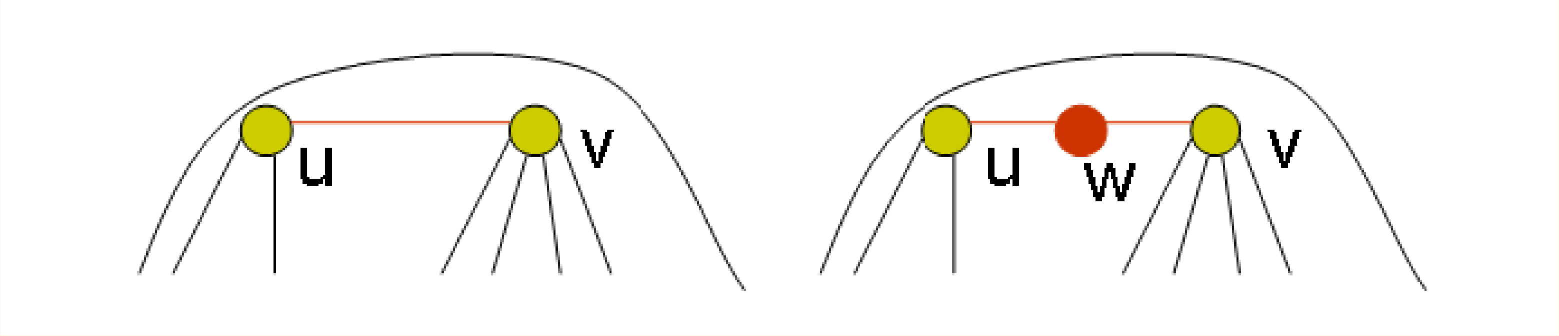
Let u be a leaf of G. If $G - u$ is acyclically Δ_0-edge choosable, so is G and $a_{list}'(G) = \max\{a_{list}'(G - u), \Delta(G)\}$.



If G is a tree, then G is acyclically Δ_0-edge choosable and $a_{list}'(G) = \Delta(G)$.

Subdividing an edge

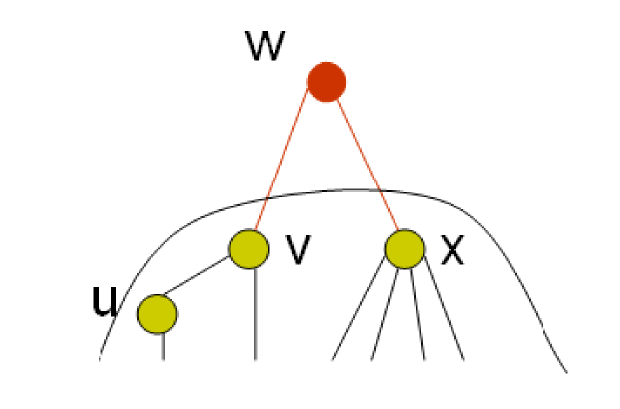
If G is obtained from an acyclically $(\Delta_1 + 1)$-edge choosable graph H by subdividing an edge, then G is $(\Delta_1 + 1)$-edge choosable.



If C is a cycle, then $a_{list}'(C) = 3$.

Adding a path

Let deg(w) = 2 in G. Let $|L(e)| \geq \Delta_0(e) + 1$ for each edge e containing w. Suppose that $G - w$ has an acyclic L-edge coloring φ, then φ can be extended to an acyclic L-edge coloring of G if all of the following conditions hold.

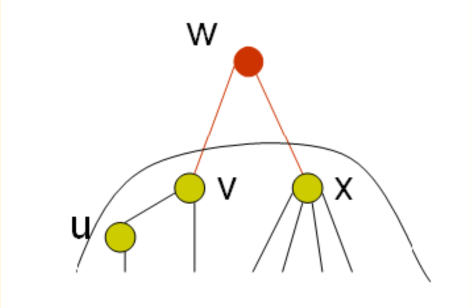


Adding a path

1. $\deg(v) + \deg(x) \leq |L(wx)| + 3$.

2. If $\deg(v) + \deg(x) = |L(wx)| + 3$, then v and x are adjacent.

3. If $\deg(v) + \deg(x) \geq |L(wx)| + 2$, then $\deg(u) + \deg(v) \leq |L(vw)| + 1$ for some neighbor $u \neq w$ of v.



Adding a path

Assume that u and x are distinct vertices of a graph H and G is obtained from H by adding a new path of length at least 3 connecting u and x. If H is acyclically $(\Delta_0 + 1)$-edge choosable, so is G.

If $a_{\text{list}}'(H) \leq \Delta_0(H) + 1$, then $a_{\text{list}}'(G) \leq \Delta_0(G) + 1$.

Adding a path

Assume that u and y are two vertices of a graph H and G is obtained from H by adding a new path of length at least 4 connecting u and y and $\Delta(G) \geq 3$.

If $a_{\text{list}}'(H) = \Delta_0(H)$, then $a_{\text{list}}'(G) = \Delta_0(G)$.

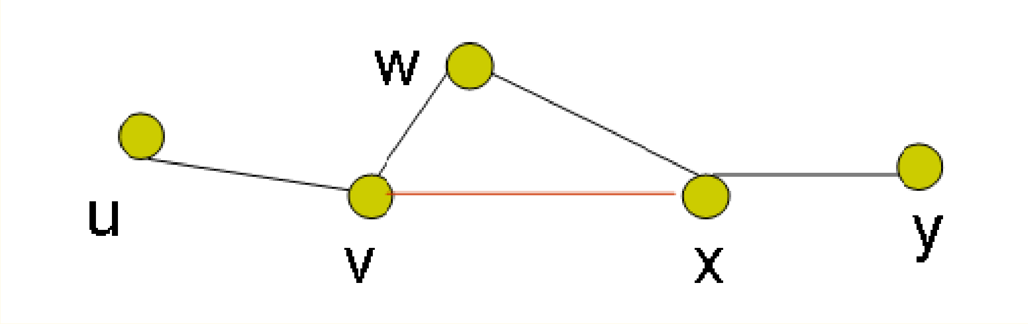
Joining two vertices of degree 2 (type A)

Let H be a graph and $uvwxy$ be a path of H in which v and x are non-adjacent vertices of degree 2 and w is their unique common neighbor. Let G be a graph obtained from H by adding a new edge joining v and x. Let L be an edge-list of G such that $|L(e)| \geq \Delta_0(e) + 1$ for $e \in \{uv, vw, wx, xy\}$ and $|L(vx)| \geq \Delta_1(vx) + 1$.

If H has an acyclic L-edge coloring, then G has an acyclic L-edge coloring under some conditions:

(i) $\max\{\deg(u), \deg(w), \deg(y)\} \leq 3$;

(ii) at most one of u and y is of degree $\Delta(H)$.



Joining two vertices of degree 2 (type A)

Let *H* be a graph with $\Delta(H) \leq 3$. Suppose that *v* and *x* are non-adjacent vertices of degree 2 and *w* is their unique common neighbor. Let *G* be a graph obtained from *H* by adding a new edge joining *v* and *x*. If *H* is acyclically $(\Delta_0 + 1)$-edge choosable, then *G* is $(\Delta_0 + 1)$-edge choosable.

Joining two vertices of degree 2 (type A)

Let H be a graph with $\Delta(H) \geq 4$.
Suppose that v and x are non-adjacent vertices of degree 2 and w is their unique common neighbor. Let G be a graph obtained from H by adding a new edge joining v and x, If H is acyclically $(\Delta_1 + 1)$-edge choosable, then G is $(\Delta_1 + 1)$-edge choosable under some conditions.

Joining two vertices of degree 2 (type B)

Let *H* be a graph with two non-adjacent vertices *v* and *x* of degree 2 such that each of them has a neighbor of degree at most 3. Let *G* be a graph obtained from *H* by adding a new edge joining *v* and *x*. Let *L* be an edge-list of *G* such that $|L(e)| \geq \max\{\Delta_1(e) + 1, 5\}$ for any edge *e* incident with *v* or *x*. If *H* has an acyclic *L*-edge coloring, so does *G*.

Joining two vertices of degree 2 (type B)

Let H be a graph with two non-adjacent vertices v and x of degree 2 such that each of them has a neighbor of degree at most 3. Let G be a graph obtained from H by adding a new edge joining v and x.

If H is $(\Delta_1 + 1)$-edge choosable, then G is $\max\{\Delta_1 + 1, 5\}$-edge choosable.

If $a_{list}'(H) \leq \Delta(H) + 1$, then $a_{list}'(G) \leq \Delta(G) + 1$.

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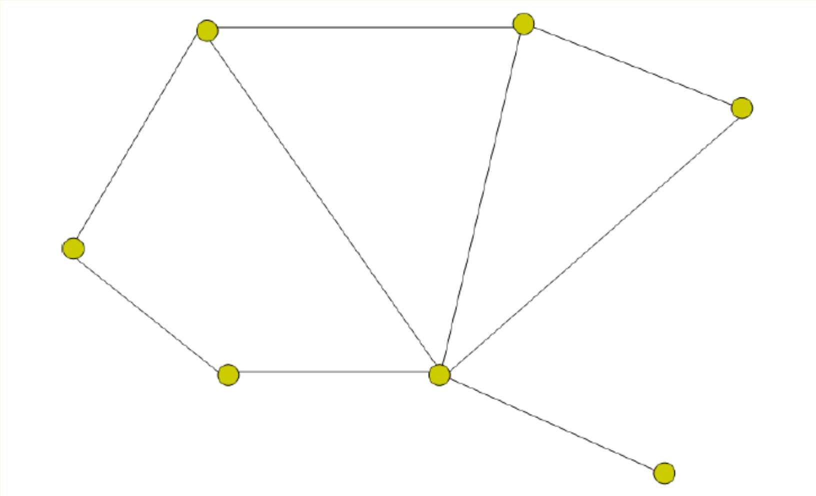
Outerplanar graphs

A graph is outerplanar if it has an embedding in the Euclidean plane such that every vertex lies on the unbounded face.



Outerplanar graphs

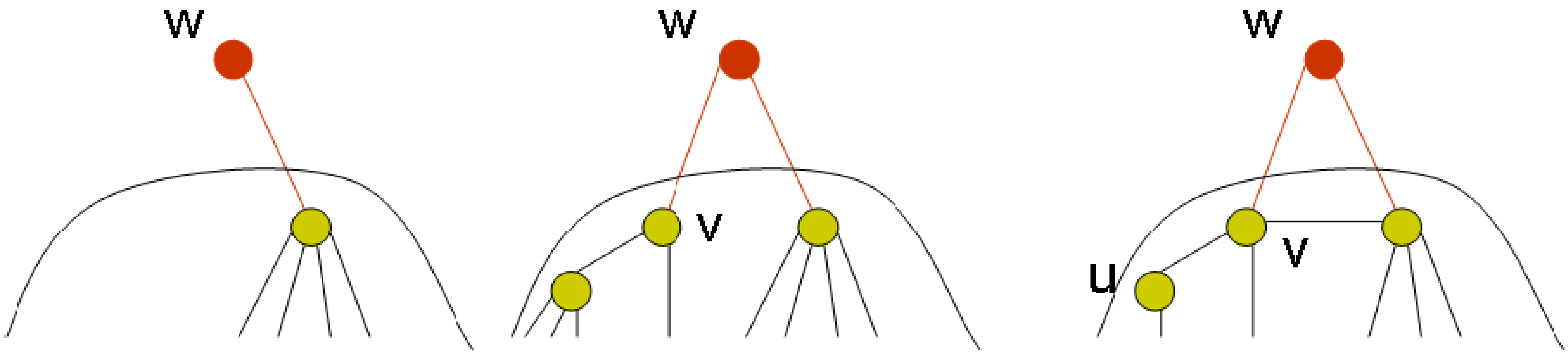
(Hackmann, Kemnitz, 2001) Let G be an outerplanar graph.

(i) there exists a leaf w, or

(ii) there exists an edge vw such that $\deg(v) \leq 3$ and $\deg(w) = 2$, or

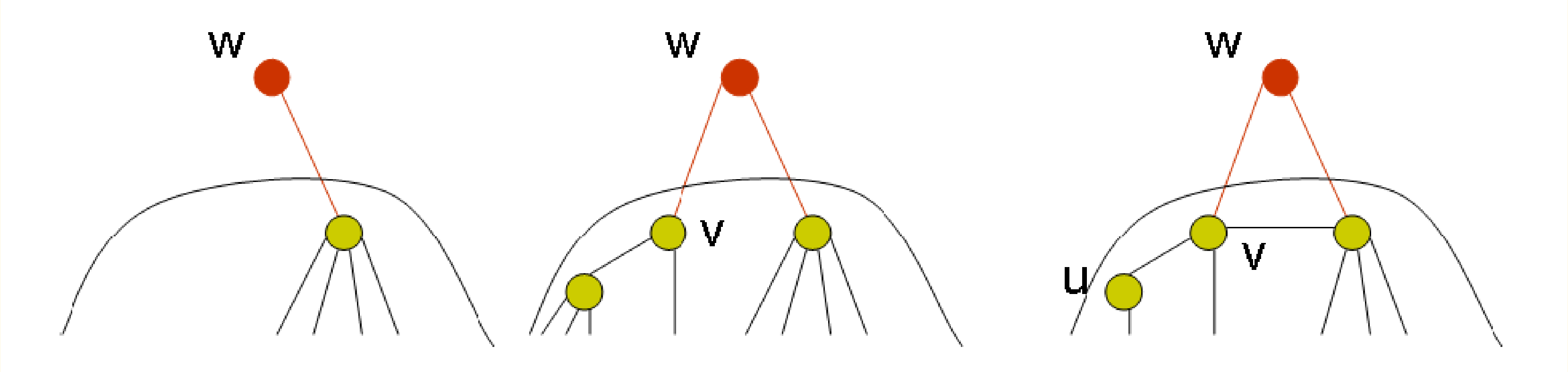
(iii) there exists edges uv and vw such that $\deg(u) = 2$, $\deg(v) = 4$, and $\deg(w) = 2$.

Outerplanar graphs



Outerplanar graphs

If G is an outerplanar graph, then G is acyclically $(\Delta_0 + 1)$-edge choosable and $a_{list}'(G) \leq \Delta(G) + 1$.



Outerplanar graphs

If G is an outerplanar graph with at most one face of length 3 or 4, then G is acyclically $\max\{\Delta_0, 3\}$-edge choosable. When $\Delta(G) \geq 3$, $a_{list}'(G) = \Delta(G)$.

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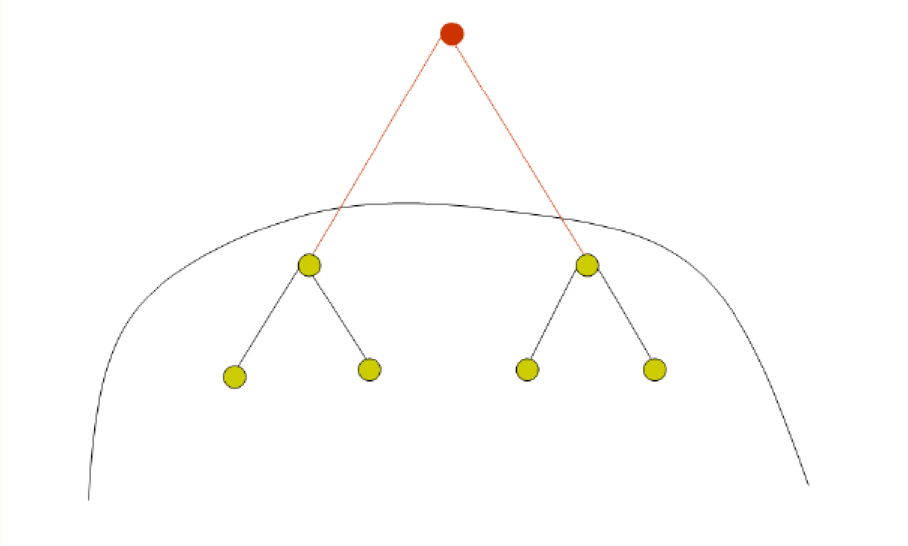
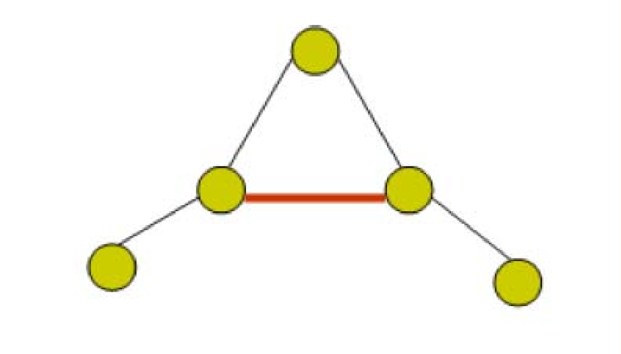
Non-regular subcubic graphs

If a connected graph G satisfies $\Delta(G) \leq 3$ and $\delta(G) \leq 2$, then G is acyclically $(\Delta_0 + 1)$-edge choosable and $a_{list}'(G) \leq \Delta(G) + 1$.



Cubic graphs with triangles

If $G \neq K_4$ is a connected cubic graph with girth 3, then G is acyclically $(\Delta_0 + 1)$-edge choosable and $a_{\text{list}}'(G) \leq \Delta(G) + 1$.



If G is a cubic graph, then $a_{\text{list}}'(G) \leq 5$.

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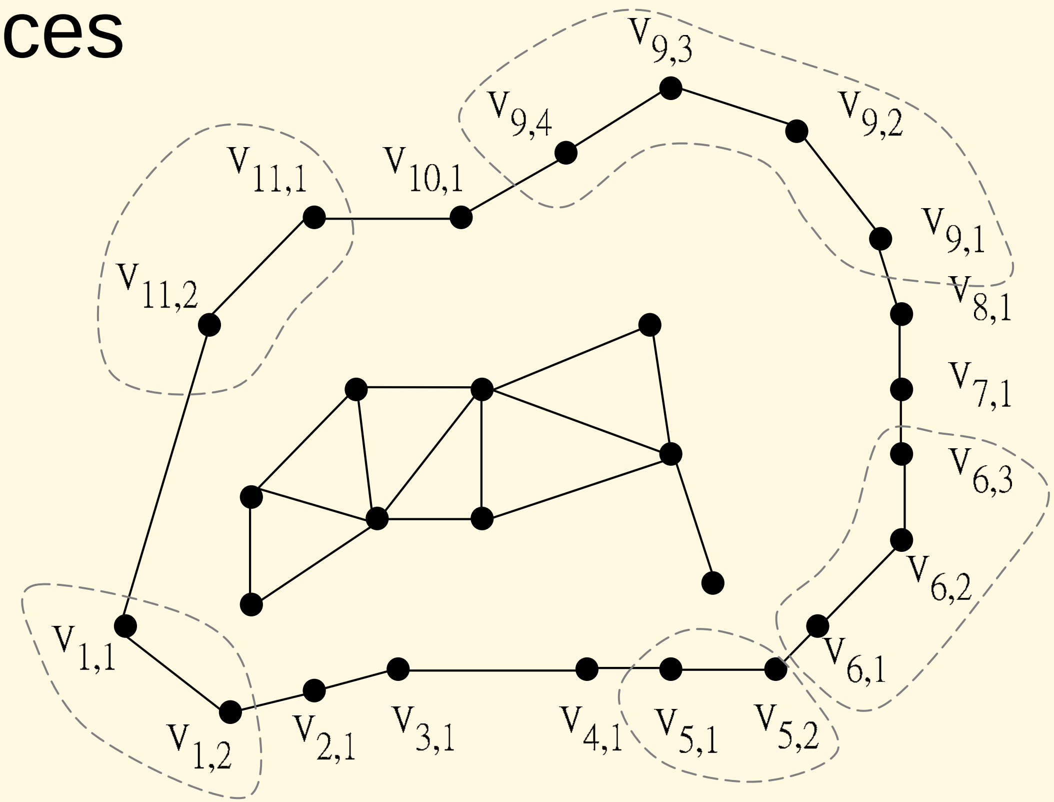
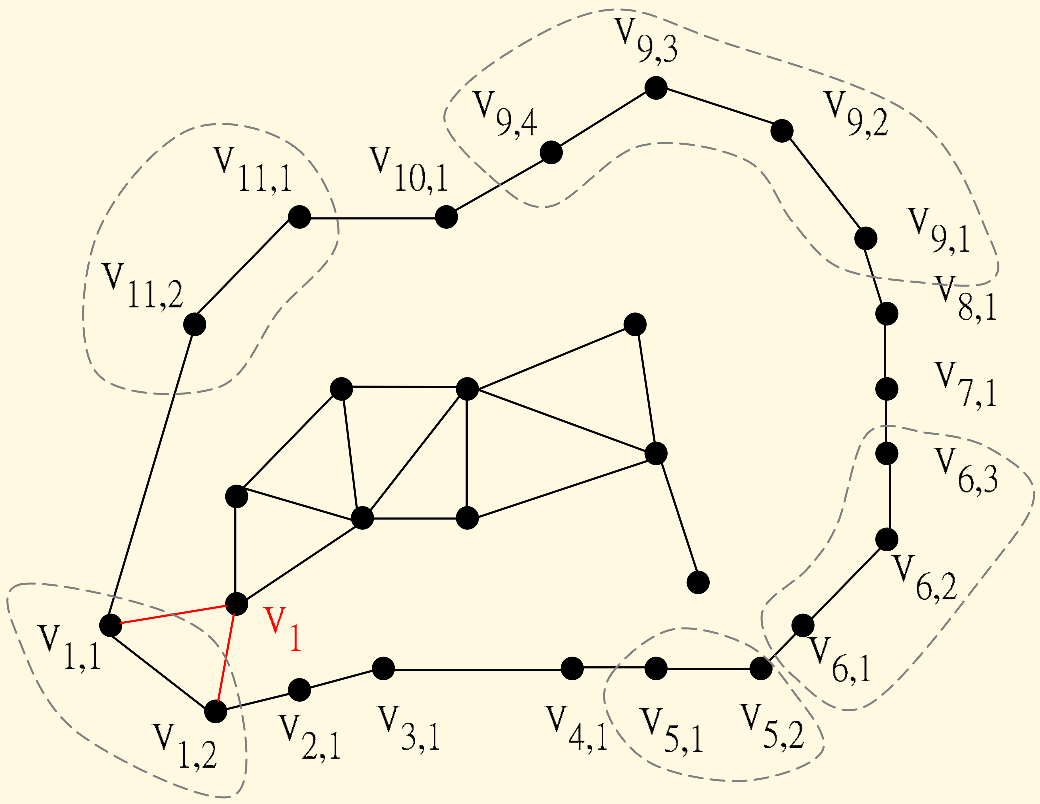
Attaching a cycle of type (2,1,1,1,2,3,1,1,4,1,2)

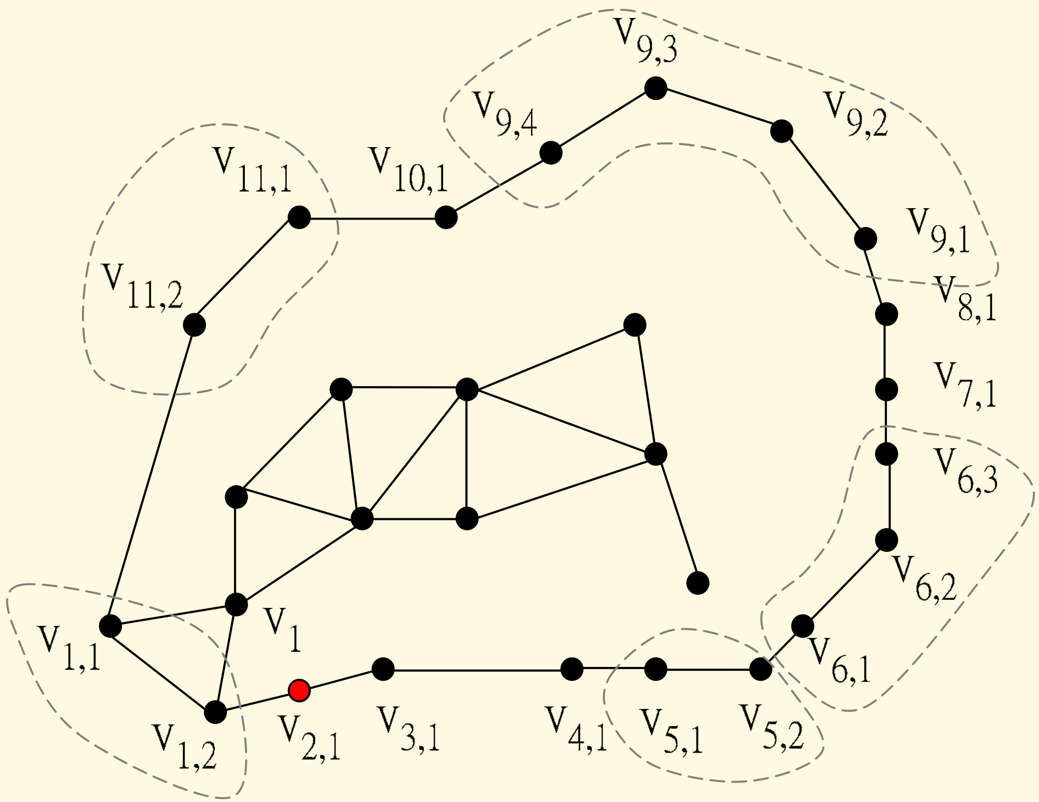
$2+1+1+1+2+3+1+1+4+1+2$
=19 vertices



Attaching a cycle of type (2,1,1,1,2,3,1,1,4,1,2)



Attaching a cycle of type (2,1,1,1,2,3,1,1,4,1,2)



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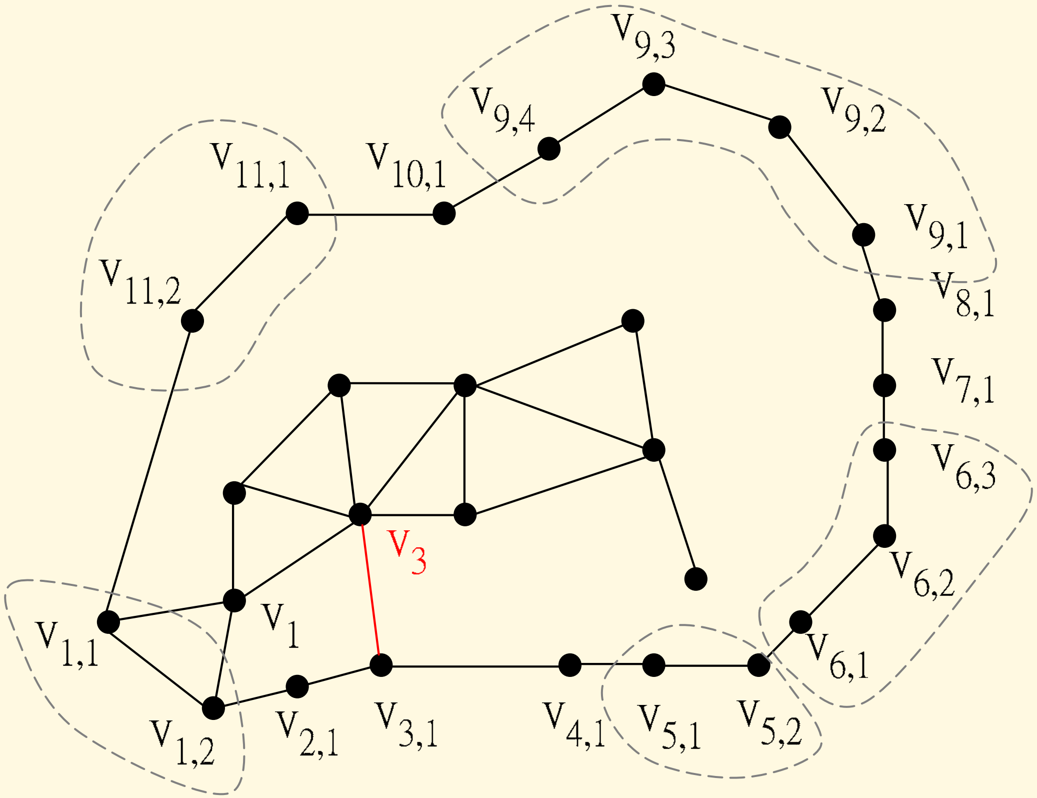
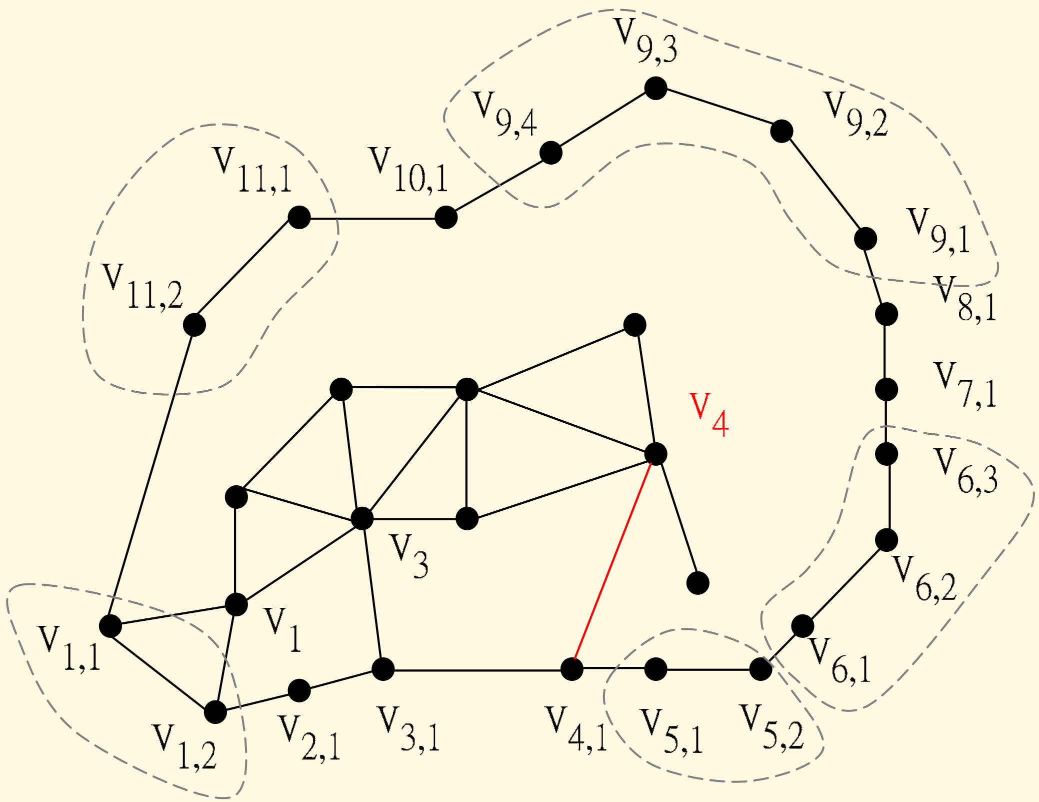
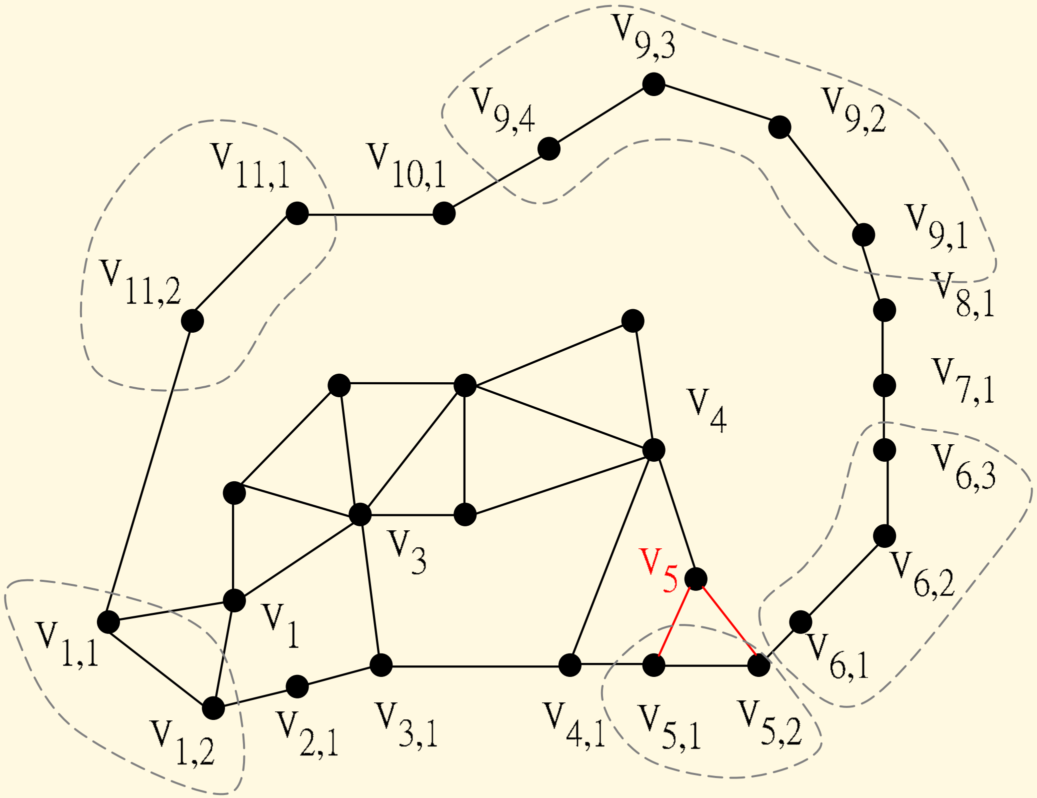


Attaching a cycle of type (2,1,1,1,2,3,1,1,4,1,2)



Attaching a cycle of type (2,1,1,1,2,3,1,1,4,1,2)



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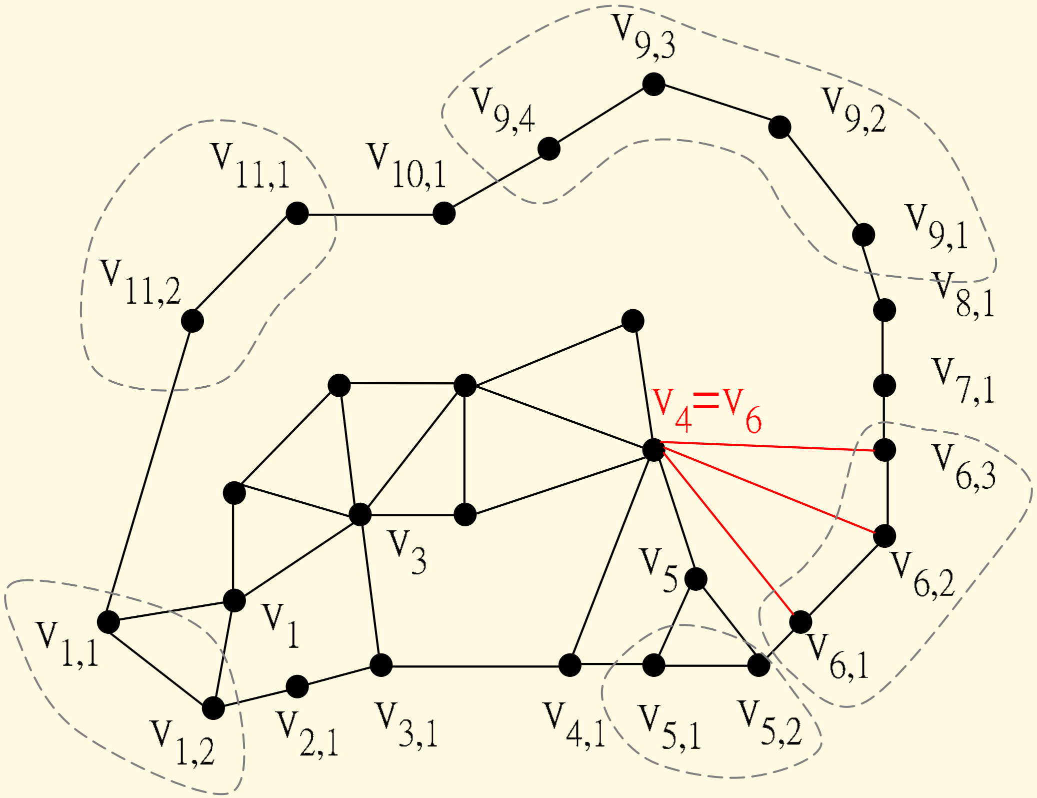
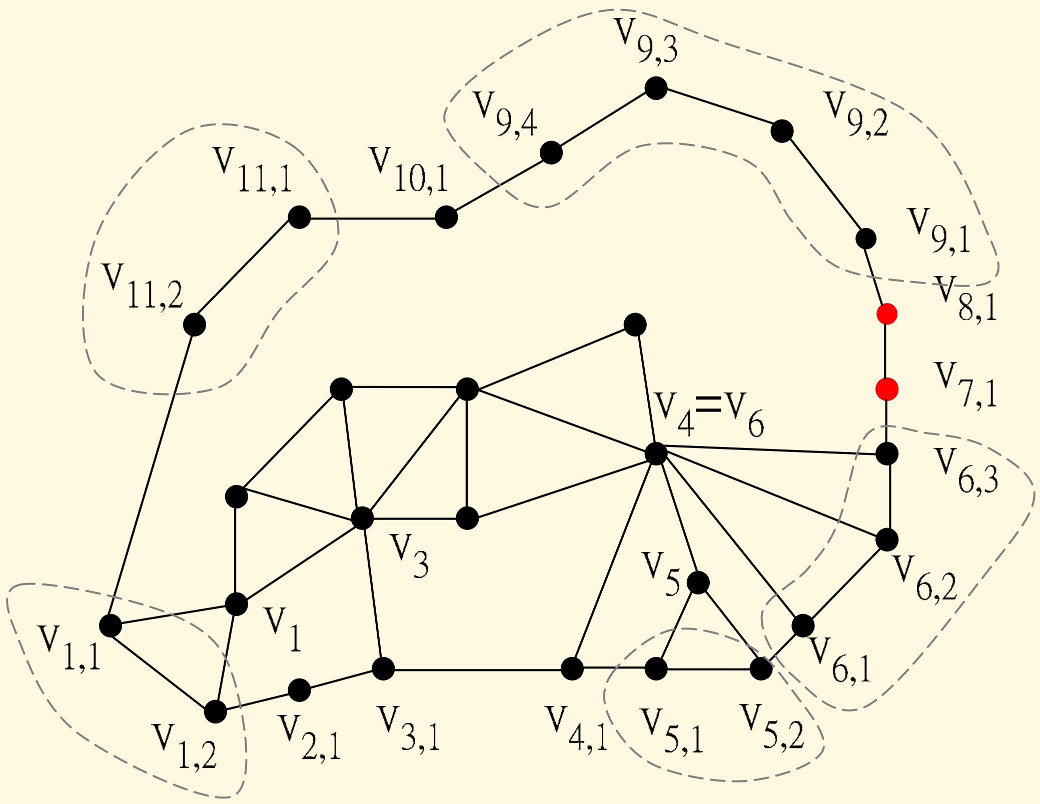


Attaching a cycle of type (2,1,1,1,2,3,1,1,4,1,2)



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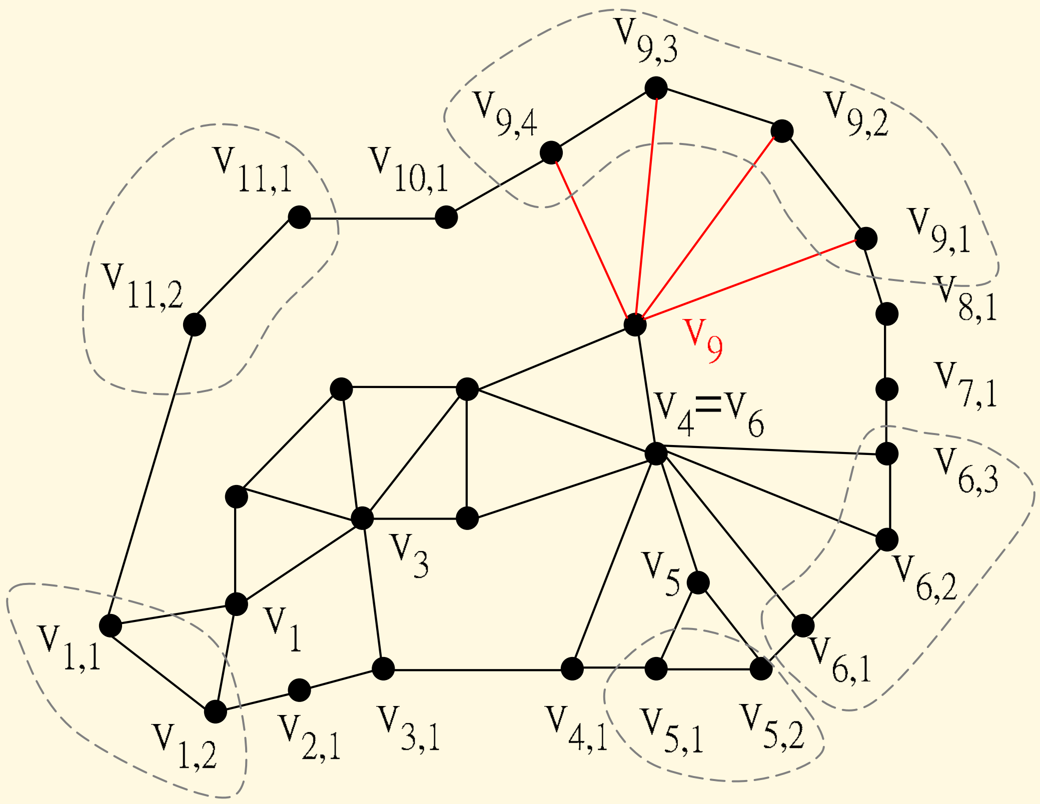
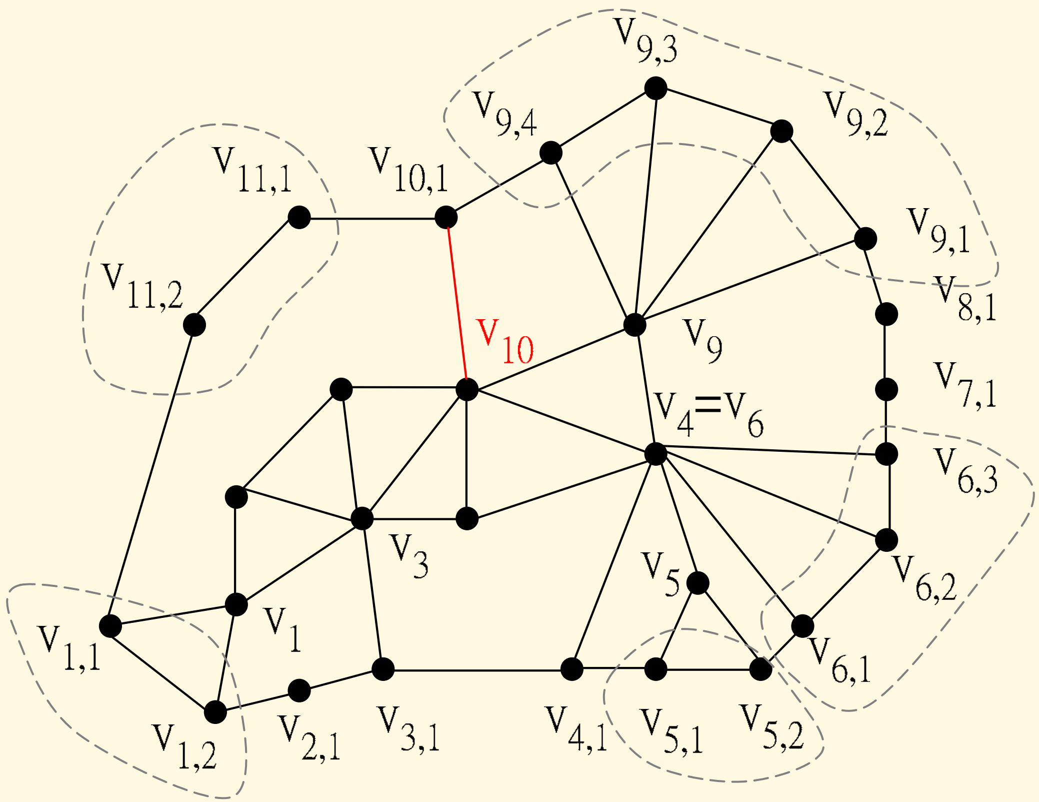
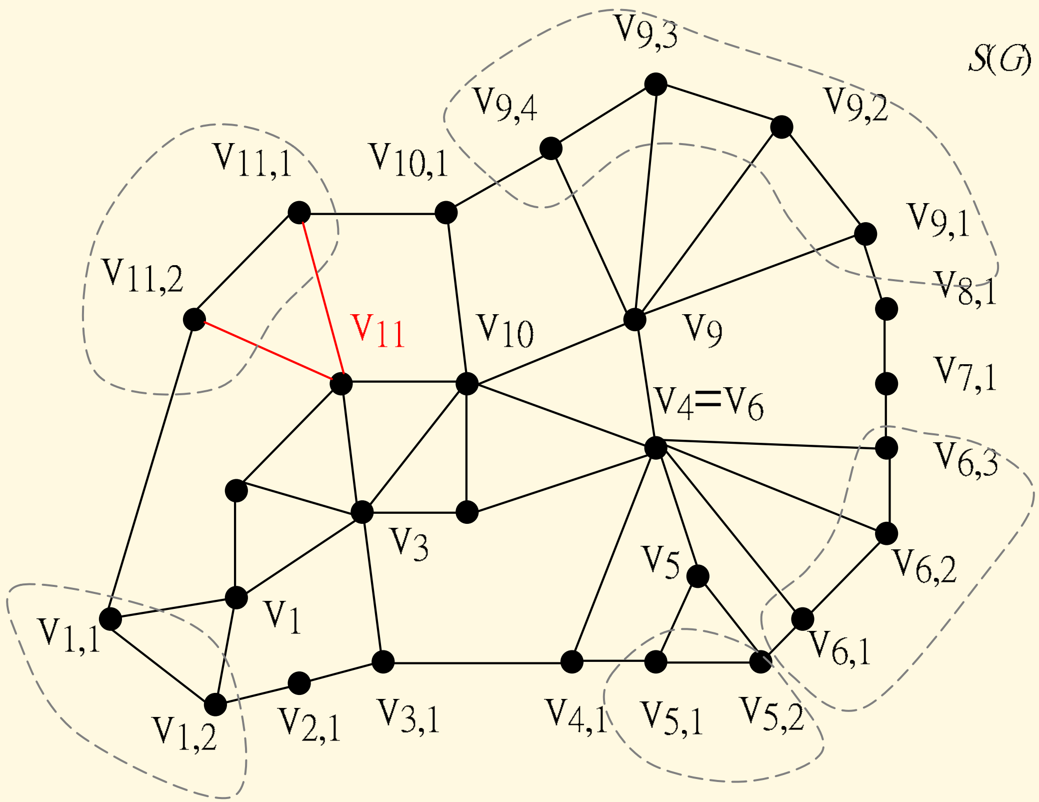


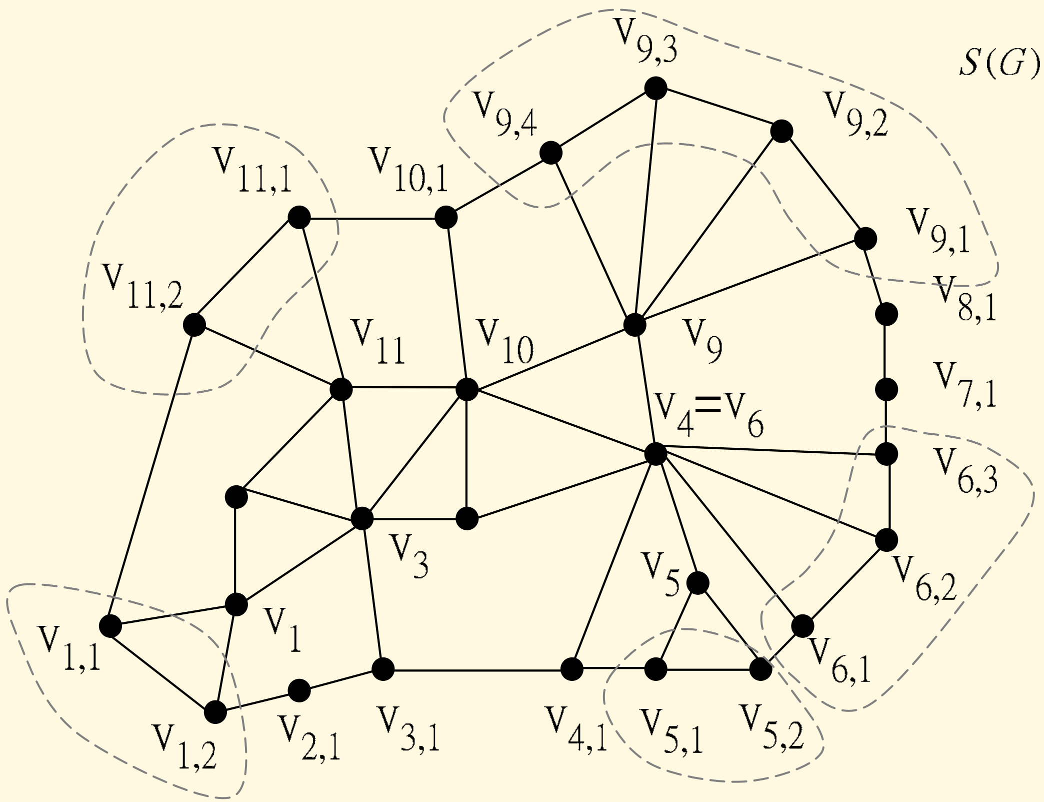
Attaching a cycle of type (2,1,1,1,2,3,1,1,4,1,2)



Attaching a cycle of type (2,1,1,1,2,3,1,1,4,1,2)



Attaching a cycle of type (2,1,1,1,2,3,1,1,4,1,2)



Attaching a cycle

Assume that $S(G)$ is a graph obtained from G by attaching a cycle of type $(l_1, l_2, \ldots, l_k)$, where $\sum_{i=1}^{k} l_i \geq 4$. Let $l_i \geq 2$ for some i or let $v_{j,1}$ have no neighbor in G for some j.

If G is acyclically $\max\{\Delta_1 + 1, 5\}$-edge choosable, so is $S(G)$.

If $a_{\text{list}}'(G) \leq \max\{\Delta(G) + 1, 5\}$, then $a_{\text{list}}'(S(G)) \leq \max\{\Delta(S(G)) + 1, 5\}$,

Attaching a cycle

Assume that $S(G)$ is a graph obtained from G by attaching a cycle of type $(l_1, l_2, \ldots, l_k)$, where $\sum_{i=1}^{k} l_i \geq 4$. Let $l_i \geq 3$ for some i and $l_j \geq 2$ for each j. If G is acyclically Δ_0-edge choosable, then $S(G)$ is acyclically $\max\{\Delta_0, 6\}$-edge choosable.

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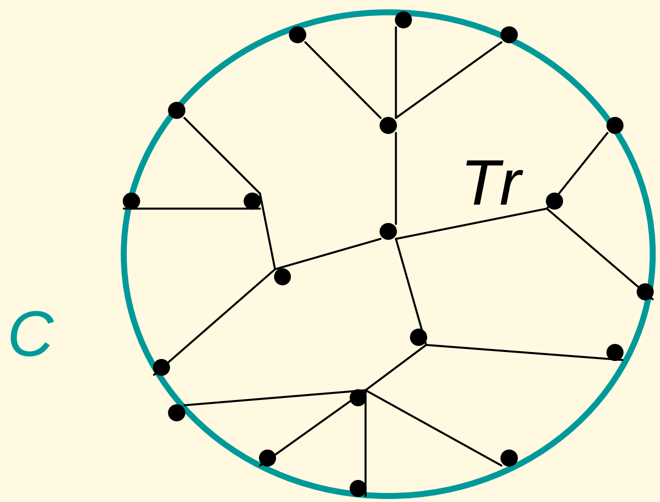
Halin Graphs

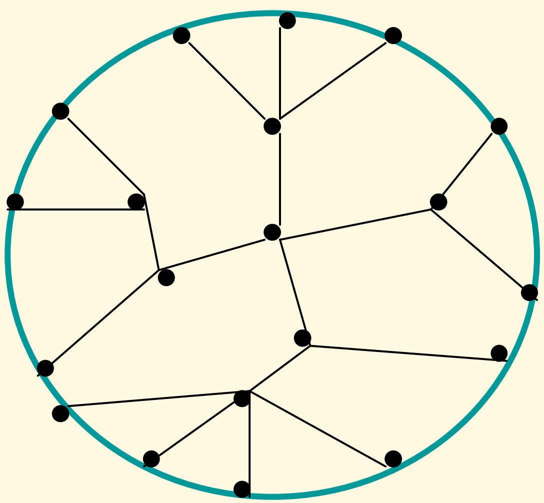
A Halin graph *H* is a plane graph obtained by drawing a tree *Tr* in the plane, where *Tr* has no vertex of degree 2, and a cycle *C* through all leaves of *Tr* in the plane.



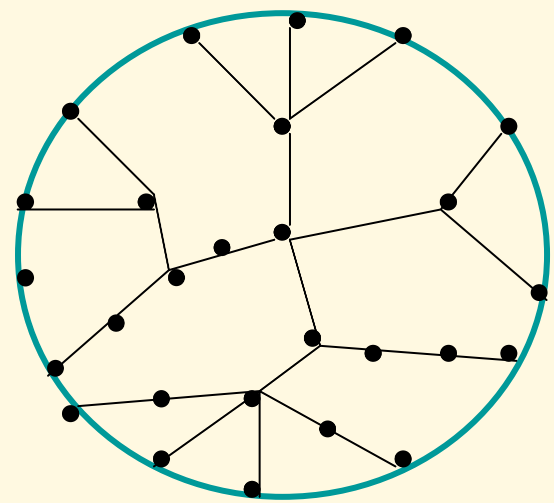
Subdivisions

A graph *H* is called a subdivision of a graph *G* if *H* can be obtained from *G* by inserting new vertices in edges of *G*.

G



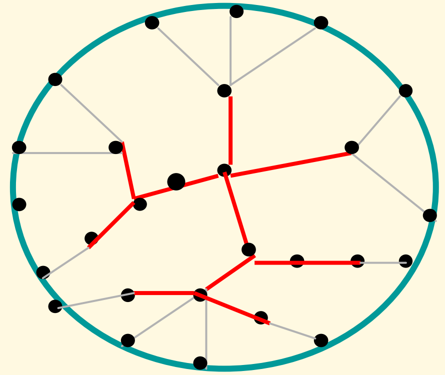
H



Subdivisions of Halin graphs

If $G = Tr^* \cup C^*$ is a subdivision of a Halin graph $H = Tr \cup C$ and $G \neq K_4$, then G acyclically $(\Delta_1 + 1)$-edge choosable and $a_{list}'(G) \leq \Delta(G) + 1$.

*Tr**



Halin graphs

If $H = Tr \cup C$ is Halin graph that contains two triangles sharing a common edge, the H is acyclically $\max\{\Delta_0, 6\}$-edge choosable. In particular, $a_{list}'(H) = \Delta(H)$ when $\Delta(H) \geq 6$.

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List Coloring Conjecture

For any graph G, $\chi_{\text{list}}'(G) = \chi'(G)$.

Open problem 1

Does $a_{\text{list}}'(G) = a'(G)$ hold for every graph G?

Open problem 2

Does $a_{\text{list}}'(G) \leq \Delta(G) + 2$ hold for every graph G?

Stronger forms:

Is G acyclically $(\Delta_1 + 2)$-edge choosable for any G?

Is G acyclically $(\Delta_0 + 2)$-edge choosable for any G?