Extension of Fouquet-Jolivet’s Conjecture

Zhiquan Hu

Faculty of Math. and Stat.
Central China Normal University
Wuhan 430079, PRC

Joint work with
Guantao Chen and Yaping Wu

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Dedicated to Professor Tian’s 70th birthday
Notations and Definitions

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- $\alpha(G)$: the independence number of $G$
- $c(G)$: the circumference of $G$
Classic results for hamiltonian graphs

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- **Theorem C (Chvátal & Erdős, 1972)** Let $G$ be a $k$-connected graph of order $n \geq 3$ and $\alpha$ be the independence number of $G$. If $\alpha \leq \kappa$, then $G$ is hamiltonian.
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- **Theorem D (Fan, 1984)** Let $G$ be a $k$-connected graph. If $\mu(G) \geq n/2$, then $G$ is hamiltonian.
Long Cycles Involving Independence Numbers

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- **Conjecture 2 (Fouquet & Jolivet, 1978).** Let $G$ be a $k$-connected graph of order $n$. If $\alpha \geq k \geq 2$, then $c(G) \geq \frac{k(n+\alpha-k)}{\alpha}$.
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- **Progress of Fouquet-Jolivet’s Conjecture:**
  - True for $k = \alpha - 1, \alpha - 2$ (Fournier, 1982)
  - True for $k = 2$ (Fournier, 1984)
  - True for $k = 3$ (Manoussakis, Graphs and Combinators 2009)
Main Results

- Theorem 3 (Chen, Hu & Wu, 2008) Let $G$ be a 4-connected graph of order $n$ with independence number $\alpha$. If $\alpha \geq 4$. Then,

$$c(G) \geq \frac{4(n+\alpha-4)}{\alpha}.$$  

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- **Question (asked by a referee of JGT)**

  - Whether it is possible to prove a weaker result like

\[
c(G) \geq \frac{k(n+\alpha-k)}{\alpha} - c_k
\]

  for a constant $c_k$?

- **Theorem 4 (Chen, Hu & Wu, 2009)** Let $G$ be a $k$-connected graph of order $n$ and independence number $\alpha$. If $\alpha \geq k \geq 4$, then

\[
c(G) \geq \frac{k(n+\alpha-k)}{\alpha} - \frac{(k-3)(k-4)}{2}.
\]
Theorem 5 (Chen, Hu & Wu, 2009) Let $G$ be a $k$-connected graph, $k \geq 2$, of order $n$ and independence number $\alpha$. If $k \leq \alpha \leq k + 3$, then $c(G) \geq \frac{k(n+\alpha-k)}{\alpha}$.

The conjecture of Fouquet and Jolivet is true for $k = \alpha - 3$. 

In order to prove Theorems 4 and 5, we proved a key lemma on how inserting vertices into a cycle and proposed a conjecture on the structure of graphs with given independent number (Conjecture 17).

We proved Conjecture 17 on March 2010 and get the following two results:
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Theorem 6 (Chen, Hu & Wu, March 2010) Let $G$ be a $k$-connected graph with $k \geq 2$, let $C$ be a cycle of $G$ and let $H$ be any induced subgraph of $G - V(C)$. Then for any real number $s \geq 1$,

$$c(G) \geq \min\{ks, |C| + |H| - \alpha(H)(s - 1)\},$$
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Theorem 7 (Chen, Hu & Wu, March 2010) Let \( G \) be a \( k \)-connected graph, \( k \geq 2 \), of order \( n \) and independence number \( \alpha \). If \( \alpha \geq k \), then

\[
c(G) \geq \frac{k(n + \alpha - k)}{\alpha} - (p - 1)(k - k_0),
\]

where \( p = \lfloor \frac{\alpha}{k} \rfloor \) and \( k_0 \) is an integer such that Conjecture 8 is true for \( k = k_0 \).
Conjecture 8 (J. Chen, L. Chen, and D. Liu) Let $G$ be a $k$-connected graph and $k \geq 2$. Then, for any two cycles $C_1$ and $C_2$ in $G$, there exist two cycles $C_1^*$ and $C_2^*$ such that $V(C_1^*) \cup V(C_2^*) \supseteq V(C_1) \cup V(C_2)$ and $|V(C_1^*) \cap V(C_2^*)| \geq k$. 
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$$c(G) \geq \frac{k(n + \alpha - k)}{\alpha}.$$
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Theorem 12 (Chen, Hu & Wu, 2010) Let $G$ be a $k$-connected graph of order $n$ and independence number $\alpha$. If $\alpha \geq k \geq 2$, then

$$c(G) \geq \min \left\{ n, \max \left\{ \frac{k(n + \alpha - k)}{\alpha}, k \left[ \frac{n + 2\alpha - 2k}{\alpha} \right] \right\} \right\}.$$
Example. Let $G := K_k + (kK_p \cup mK_{p-1})$, where $k, p \geq 2$ and $m \geq 1$.

$\blacklozenge$ $n = k(p + 1) + m(p - 1)$, $\kappa = k$, and $\alpha = k + m$.

$\blacklozenge$ $c(G) = k(p + 1) = k \left\lfloor \frac{n+2\alpha-2k}{\alpha} \right\rfloor$.

$\blacklozenge$ $c(G) - \frac{k(n+\alpha-k)}{\alpha} = k[(p + 1) - \frac{k(p+1)+mp}{k+m}] = \frac{km}{k+m} \to k \ (m \to \infty)$.

Therefore, the low bound in Theorem 12 is sharp and better than that of Theorem 10.

Figure: $c(G) > k \left\lfloor \frac{n+2\alpha-2k}{\alpha} \right\rfloor > \frac{k(n+\alpha-k)}{\alpha}$
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Remark 1: The function
\[
f(G) := \max\left\{ \frac{k(|G| + \alpha(G) - k)}{\alpha(G)}, k \left\lfloor \frac{|G| + 2\alpha(G) - 2k}{\alpha(G)} \right\rfloor \right\}
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Remark 2: If $\emptyset \neq H \subseteq G$, then $f(G[V(H)]) \geq f(H)$.
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**Example.** $c(G) = k(p+1) = \min\{|H|, f(H)\} > f(G)$. 

Extension of Fouquet-Jolivet's Conjecture
Let $G$ be a $k$-connected graph and let $V_0$ a given subset of $V(G)$. It is interesting to know that whether $V_0$ is cycliable in $G$ and if $V_0$ is not cycliable in $G$, how many vertices of $V_0$ can be contained in one common cycle of $G$. 

Theorem 15 (Shi, 1992) Let $G$ be a graph on $n$ vertices and let $W \subseteq V(G)$ such that each pair of nonadjacent vertices $u, v \in W$ satisfies $d(u) + d(v) \geq n$. If $|W| \geq 3$, then $G$ contains a cycle through all vertices of $W$. 

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Extension of Fouquet-Jolivet's Conjecture
Long Cycles Intersecting a given subgraph

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**Extension of Fouquet-Jolivet’s Conjecture**
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Theorem 16 (Chen, Hu & Wu, 2010) Let $G$ be a $k$-connected graph with $k \geq 2$ and let $H$ be a subgraph of $G$. If $|H| \geq \alpha(H) + k$, then there is a cycle $C$ in $G$ such that

$$|V(C) \cap V(H)| \geq \min \left\{ |H|, k \left[ \frac{|H| + \alpha(H) - k}{\alpha(H)} \right] \right\}.$$
Example. Let $G := G_k + (mK_p \cup tK_{p-1})$, where $k, p \geq 2$, $t \geq 1$ and $m \geq k + 1$.

For $\ell \leq t$, $H := mK_p \cup \ell K_{p-1}$ is a subgraph of $G$ with

$$k \times \left\lfloor \frac{|H| + \alpha(H) - k}{\alpha(H)} \right\rfloor = k \times \left\lfloor \frac{mp + \ell(p - 1) + (m + \ell) - k}{m + \ell} \right\rfloor = kp,$$

which is the maximum number of vertices of $H$ that can be contained in a common cycle of $G$. Therefore, the low bound in Theorem 16 is sharp.
Ideas of Proofs

1° Establish some lemmas on inserting $H$-vertices into a cycle $C$ of $G - V(H)$.

2° Study the structure of graphs with given independence number.

3° Establish a low bound of $c(G)$ relative to a cycle $C$ and an induced subgraph $H$ of $G - C$.

4° By using 3° and Kouider’s Theorem to get the desired bound.
Ideas of Proofs

1° Establish some lemmas on inserting $H$-vertices into a cycle $C$ of $G - V(H)$.

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4° By using 3° and Kouider’s Theorem to get the desired bound.

Theorem (Kouider, JCTB 1994) Let $G$ be a $k$-connected graph, $k \geq 2$, of order $n$ and $H$ be an induced subgraph of $G$. Then, either the vertices of $H$ are covered by one cycle of $G$ or else $G$ has a cycle $C$ satisfying $\alpha(H - V(C)) \leq \alpha(H) - k$. 

Zhiquan Hu Extension of Fouquet-Jolivet’s Conjecture
Inserting $H$-vertices into a cycle $C$ of $G - V(H)$

**Definition** Let $C$ be a cycle of $G$ and let $H$ be an induced subgraph of $G - V(C)$. For $x_1 \neq x_2 \in V(C)$, the segment $C[x_1, x_2]$ is called a normal $H$-interval of $C$ if there exist two internally vertex disjoint paths $P_1, P_2$ in $G$ from $V(H)$ to $V(C)$ such that

(N-1) $V(P_i) \cap V(C) = \{x_i\}$, for each $i = 1, 2$ and

(N-2) $|V(H) \cap (V(P_1) \cup V(P_2))| = \min \{|H|, 2\}$. 
Lemma (Hu, Tian & Wei, JCT B82 (2001)) Let $m \geq 0$ and $k \geq 2$. Let $G$ be a $(m + k)$-connected graph and $M$ an $m$-matching of $G$. Let $S \subseteq V(G) - V(M)$ with $|S| \leq k - 2$ and let $C$ be a longest cycle passing through $M \cup S$. If $l(C) < \min \{|V(G)|, 2L - m\}$, where $L$ is a constant, then every component $H$ of $G - V(C)$ has a vertex $x$ with $d_G(x) < L$. 
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The essential part of the proof: If the Lemma is not true, then there exists a set of $(m + k)$ pairwise edge disjoint normal $H$-intervals of $C$. 
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The essential part of the proof: If the Lemma is not true, then there exists a set of $(m + k)$ pairwise edge disjoint normal $H$-intervals of $C$.

Question: What happens if $m = 0$ and $H$ is an induced subgraph of $G - V(C)$?
Inserting vertices into a cycle —a key lemma

▶ Definition Let $V_0 \subseteq V(G)$. A cycle $C$ of $G$ is called a maximal $V_0$-cycle if there is no cycle $C'$ in $G$ such that $V(C) \cap V_0$ is a proper subset of $V(C') \cap V_0$. 
Inserting vertices into a cycle — a key lemma

Definition Let $V_0 \subseteq V(G)$. A cycle $C$ of $G$ is called a maximal $V_0$-cycle if there is no cycle $C'$ in $G$ such that $V(C) \cap V_0$ is a proper subset of $V(C') \cap V_0$.

Key Lemma. Let $G$ be a $k$-connected graph, $k \geq 2$ and $s \geq 1$ be two integers. Let $V_0$ be a subset of $V(G)$, let $C$ be a maximal $V_0$-cycle in $G$ with length at least $k$ and let $H$ be a subgraph of $G[V_0 - V(C)]$ with $|H| \geq s$. If every normal $H$-interval $C[x_1, x_2]$ of $C$ satisfies $|C(x_1, x_2) \cap V_0| \geq s$, then

(i) $|V(C) \cap V_0| \geq ks$, and

(ii) $|V(C)| \geq k(s + 1)$. 
How to use the Key Lemma?

By taking \( V_0 = V(G) \) in the Key Lemma, we see that if \( C \) is a maximal cycle in \( G \) with \( |C| < k(s + 1) \), then for every induced subgraph \( H \) of \( G - V(C) \) with \( |H| \geq s \), there is a normal \( H \)-interval \( C[x_1,x_2] \) such that \( |C(x_1,x_2)| \leq s - 1 \).
How to use the Key Lemma?

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Inserting vertices of $H$ to $C$ by removing paths—Find a cycle $C'$ with $V(C') \supseteq C[x_2, x_1]$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$. 

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Zhiquan Hu  
Extension of Fouquet-Jolivet's Conjecture
Structures of graphs with independence number $\alpha$

- Let $G$ be a graph with independence number $\alpha$.
  
  - $\alpha = 1$: $G$ is Hamilton-connected.
  
  - $\alpha = 2$: either $G$ has a hamiltonian cycle or $V(G)$ has a partition $(V_1, V_2)$ such that both $G[V_1]$ and $G[V_2]$ are cliques.
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- Equivalent form:
  - $\alpha = 1$: $\forall u \neq v \in V(G)$, $G$ has a $(u, v)$-path $P$ such that
    $$\alpha(G - V(P)) \leq \alpha(G) - 1.$$  
  - $\alpha = 2$: either $G$ has a hamiltonian cycle or $V(G)$ has a partition $(V_1, V_2)$ such that
    $$\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2]).$$
Which structure of $G - V(C)$ is useful?

- **Problem (asked by a referee of JGT)**
  - Whether it is possible to prove a weaker result like
    \[
    c(G) \geq \frac{k(n + \alpha - k)}{\alpha} - c_k
    \]
    for a constant $c_k$?
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- **Remark**: The fact that “$G - V(C)$ has a large subgraph $H$ that is hamiltonian” is not useful for the conclusion that
  
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  $$c(G) \geq \frac{k(n + \alpha - k)}{\alpha} - c_k$$

- In order to solve the above problem, we propose the following conjecture.
Conjecture 17 (Chen, Hu & Wu, 2009): For any graph \( G \), one of the following two statements holds.

(i) For any two distinct vertices \( u, v \in V(G) \), there exists a \((u, v)\)-path \( P \) such that \( \alpha(G - V(P)) \leq \alpha(G) - 1 \).

(ii) there is a non-trivial partition \((V_1, V_2)\) of \( V(G) \) such that \( \alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2]) \).
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Example: The sun graph $S(C_t)$ doesn’t satisfies (i) and the cycle $C_{2m+1}$ doesn’t satisfies (ii).
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Failed to find a 2-connected graph $G$ that doesn't satisfies (i), we believe that (i) is true for every 2-connected graph. We prove the following strong result by induction.
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Lemma 18 (Chen, Hu & Wu, 2010): Let $G$ be a graph with independence number $\alpha$ and let $u, v$ be two distinct vertices of $G$. If $\kappa(G) \geq 2$, then $G$ contains a $(u, v)$-path $P$ such that $\alpha(G - V(P)) \leq \alpha(G - v) - 1$. 

Extension of Fouquet-Jolivet's Conjecture
Lemma 19 (Chen, Hu & Wu, 2009): Let $G$ be a graph with independence number $\alpha \geq 2$. If $\kappa(G) \leq 1$, then there is a non-trivial partition $(V_1, V_2)$ of $V(G)$ such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$. 
Lemma 19 (Chen, Hu & Wu, 2009): Let $G$ be a graph with independence number $\alpha \geq 2$. If $\kappa(G) \leq 1$, then there is a non-trivial partition $(V_1, V_2)$ of $V(G)$ such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$.

By Lemmas 18 and 19, Conjecture 17 is true. Further, we have
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By Lemmas 18 and 19, Conjecture 17 is true. Further, we have

Theorem 20 Let $G$ be a graph. Then, there exist two vertex disjoint induced subgraph $H_1$ and $H_2$ of $G$ such that

(i) $V(G) = V(H_1) \cup V(H_2)$ and $\alpha(G) = \alpha(H_1) + \alpha(H_2)$;

(ii) $H_2 \neq \emptyset$ and for any two distinct vertices $u, v \in V(H_2)$, there exists a $(u, v)$-path $P$ in $H_2$ such that $\alpha(H_2 - V(P)) \leq \alpha(H_2) - 1$. 

Zhiquan Hu

Extension of Fouquet-Jolivet's Conjecture
A low bound of $c(G)$ w.r.t. a cycle $C$ and an induced subgraph $H$ of $G - C$

- **Key Lemma.** Let $G$ be a $k$-connected graph, $k \geq 2$ and $s \geq 1$ be two integers. Let $V_0$ be a subset of $V(G)$, let $C$ be a maximal $V_0$-cycle in $G$ with length at least $k$ and let $H$ be a subgraph of $G[V_0 - V(C)]$ with $|H| > s - 1$. If there is no normal $H$-interval $C[x_1, x_2]$ of $C$ such that $|C(x_1, x_2) \cap V_0| \leq s - 1$, then (i) $|V(C) \cap V_0| \geq ks$; (ii) $|V(C)| \geq k(s + 1)$. 
A low bound of $c(G)$ w.r.t. a cycle $C$ and an induced subgraph $H$ of $G - C$

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- By using the Key Lemma and Theorem 20, we can prove...
A low bound of $c(G)$ w.r.t. a cycle $C$ and an induced subgraph $H$ of $G - C$

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- **By using the Key Lemma and Theorem 20, we can prove**

- **Theorem 21 (Chen, Hu & Wu, 2010).** Let $G$ be a $k$-connected graph with $k \geq 2$, let $C$ be cycle in $G$ and let $H$ be a subgraph of $G - V(C)$. Then, for every integer $t \geq 2$, we have

$$c(G) \geq \min\{kt, |C| + |H| - \alpha(H)(t - 2)\}.$$
Proof of Theorem 13

The following is a special form of Theorem 13.

**Theorem 22 (Chen, Hu & Wu, 2010)** Let $G$ be a $k$-connected graph, $k \geq 2$, of order $n$ and $V_0$ a nonempty subset of $V(G)$. Then

$$c(G) \geq \min \{ |V_0|, k \cdot \max \{ f_1(|V_0|), f_2(|V_0|) \} \},$$

where

$$f_i(V_0) = \frac{|V_0| + i(\alpha(G[V_0]) - k)}{\alpha(G[V_0])}, \ i = 1, 2.$$
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The following is a special form of Theorem 13.

▶ **Theorem 22 (Chen, Hu & Wu, 2010)** Let $G$ be a $k$-connected graph, $k \geq 2$, of order $n$ and $V_0$ a nonempty subset of $V(G)$. Then
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where

\[
    f_i(V_0) = \frac{|V_0| + i(\alpha(G[V_0]) - k)}{\alpha(G[V_0])}, \quad i = 1, 2.
\]

▶ **Proof of Theorem 22:**

♦ Find a cycle $C$ in $G$ such that $V_0 \subseteq V(C)$ or
\[
    \alpha(G[V_0] - V(C)) \leq \alpha(G[V_0]) - k \text{ (by using Kouider's Theorem).}
\]

♦ If $V_0 \subseteq V(C)$, then $c(G) \geq |C| \geq |V_0|$; If $V_0 \not\subseteq V(C)$, then
\[
    \alpha(G[V_0]) > k. \text{ So, } f_1(V_0) \geq 2.
\]
Proof of Theorem 22 (continue):

♦ By using Theorem 21 with $H = G[V_0 - V(C)]$, we get

$$c(G) \geq \min\{kt, |C| + |H| - \alpha(H)(t - 2)\}$$

$$\geq \min\{kt, |V_0| - (\alpha(G[V_0]) - k)(t - 2)\}, \quad (1)$$

where $t$ is an integer with $t \geq 2$.

♦ If $c(G) < kf_1(V_0)$, then by taking $t = \lceil f_1(V_0) \rceil$ in (1), we have

$$kf_1(V_0) > |V_0| - (\alpha(G[V_0]) - k)(\lceil f_1(V_0) \rceil - 2)$$

$$\geq |V_0| - (\alpha(G[V_0]) - k)(f_1(V_0) - 1),$$

which simplifies to $f_1(V_0) > \frac{|V_0| + \alpha(G[V_0]) - k}{\alpha(G[V_0])}$, a contradiction. Thus,

$$c(G) \geq kf_1(V_0). \quad (2)$$
Proof of Theorem 22 (continue):

If \( c(G) < k \lfloor f_2(V_0) \rfloor \), then by (2), \( \lfloor f_2(V_0) \rfloor > f_1(V_0) \geq 2 \). By taking \( t := \lfloor f_2(V_0) \rfloor \) in (1), we have

\[
\begin{align*}
c(G) & \geq |C| + |H| - \alpha(H)(\lfloor f_2(V_0) \rfloor - 2) \\
& \geq |V_0| - (\alpha(G[V_0]) - k)(\lfloor f_2(V_0) \rfloor - 2) \\
& = (|V_0| + 2\alpha(G[V_0]) - 2k) - (\alpha(G[V_0]) - k)\lfloor f_2(V_0) \rfloor.
\end{align*}
\]

This together with \( c(G) < k \lfloor f_2(V_0) \rfloor \) implies that

\[\lfloor f_2(V_0) \rfloor > \frac{|V_0| + 2\alpha(G[V_0]) - 2k}{\alpha(G[V_0])},\]

a contradiction. \( \square \)
Thank You Very Much For Your Attention!