# Heterochromatic $C_4$ in edge-colored triangle-free and bipartite graphs

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- Guanghui Wang, Hao Li, Yan Zhu, Guizhen Liu and Jiguo Yu, A note on heterochromatic $C_4$ in edge-colored triangle-free graphs. (submitted)

# Heterochromatic $C_4$ in edge-colored triangle-free and bipartite graphs

Let $G$ be an edge-colored graph. A subgraph $H$ of $G$ is called heterochromatic, or rainbow, or multicolored if any pair of edges in $H$ have distinct colors.

A heterochromatic cycle (path) of $G$ is a cycle (path) in which any pair of edges have distinct colors.

For a vertex $v \in V(G)$, a color neighbourhood of $v$ is defined as a set $T \subseteq N(v)$ such that the colors of the edges between $v$ and $T$ are pairwise distinct. A maximum color neighborhood $N^c(v)$ of $v$ is a color neighborhood of $v$ with maximum size.

Let $d^c(v)$, named the color degree of a vertex $v$, be defined as the maximum number of edges incident with $v$, that have distinct colors.

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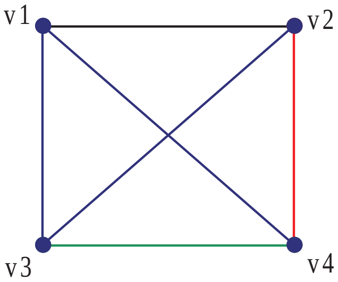

$$N^c(v_1) = \{v_2, v_3\} = \{v_2, v_4\},\ N^c(v_2) = \{v_1, v_3, v_4\},$$
$$N^c(v_3) = \{v_1, v_4\} = \{v_2, v_4\},\ N^c(v_4) = \{v_1, v_2, v_3\}.$$

$$d^c(v_1) = d^c(v_3) = 2, \quad d^c(v_2) = d^c(v_4) = 3.$$

# Heterochromatic $C_4$ in edge-colored triangle-free and bipartite graphs

Albert, Frieze and Reed (1995) showed that if $n$ is sufficiently large and the edges of the complete graph $K_n$ are colored so that no color appears more than $\lceil cn \rceil$ times, where $c < 1/32$ is a constant, then there is a heterochromatic Hamilton cycle.

Let $HC_l$ denote a heterochromatic cycle with length $l$.

**Theorem 1** (Broersma, Li, Woeginger and Zhang 2005) *Let $G$ be an edge-colored graph of order $n$ such that $c(G) \geq n$. Then $G$ contains a heterochromatic cycle of length at least $\frac{2c(G)}{n-1}$.*

**Theorem 2** (Broersma, Li, Woeginger and Zhang 2005) *Let $G$ be an edge-colored graph of order $n \geq 4$, such that $|N^c(u) \cup N^c(v)| \geq n-1$ for every pair of vertices $u$ and $v$ of $G$. Then $G$ contains at least one $HC_3$ or one $HC_4$.*

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**Theorem 3** (Li and Wang 2006) *Let* $G$ *be an edge-colored graph with order* $n$, $n \geq 3$. *If for each vertex* $v$ *of* $G$, $d^c(v) \geq \frac{n+1}{2}$, *then* $G$ *has a heterochromatic cycle.*

**Theorem 4** (Li and Wang 2006) *Let* $G$ *be an edge-colored graph with order* $n$, $n \geq 3$. *If for each* $v$ *of* $G$, $d^c(v) \geq (\frac{4\sqrt{7}}{7} - 1)n + 3 - \frac{4\sqrt{7}}{7}$, *then* $G$ *has either an* $HC_3$ *or an* $HC_4$. $(\frac{4\sqrt{7}}{7} - 1 \approx 0.515, \quad 3 - \frac{4\sqrt{7}}{7} \approx 1.488)$

**Theorem 5** (Li and Wang 2006) *Let* $G$ *be an edge-colored graph with order* $n$, $n \geq 3$. *If for each* $v$ *of* $G$, $d^c(v) \geq \frac{\sqrt{7}+1}{6}n$, *then* $G$ *has an* $HC_3$. $(\frac{\sqrt{7}+1}{6} \approx 0.608)$

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**Conjecture 6** (Li and Wang 2006) *Let $G$ be an edge-colored graph with order $n \geq 3$. If $d^c(v) \geq \frac{n+1}{2}$ for every $v \in V(G)$, then $G$ has an $HC_3$.*

They have the following example to show that if the above conjecture is true, it would be best possible. For any even integer $n$, let $B_{n/2,n/2}$ be an edge-proper-colored complete balance bipartite graph with order $n$. Then for every vertex $v$ of $B_{n/2,n/2}$ , it holds that $d^c(v) = \frac{n}{2}$, and $B_{n/2,n/2}$ has no $HC_3$.

**Theorem 7** (Li and Wang 2006) *Let $G$ be an edge-colored graph with order $n \geq 3$. If $d^c(v) \geq d \geq \frac{3n}{4} + 1$ for every $v \in V(G)$, then $G$ has an $HC_l$ such that $l \geq d - \frac{3n}{4} + 2$.*

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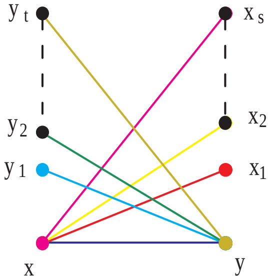
**Theorem 8** (Wang, Li, Zhu, Liu and Yu, 2009) *Let $G$ be an edge-colored triangle-free graph of order $n$, $n \geq 9$. If for each vertex $v$ of $G$, $d^c(v) \geq \frac{(3-\sqrt{5})n}{2} + 1$, then $G$ has an $HC_4$.* $(\frac{3-\sqrt{5}}{2} \approx 0.382)$

**Theorem 9** (Wang, Li, Zhu, Liu and Yu, 2009) *Let $G$ be a bipartite edge-colored graph of order $n$. If for each vertex $v$ of $G$, $d^c(v) \geq \frac{(\sqrt{5}-1)n}{4} + 1$, then $G$ has an $HC_4$.* $(\frac{(\sqrt{5}-1)}{4} \approx 0.309)$

# Heterochromatic $C_4$ in edge-colored triangle-free and bipartite graphs

Outline of proof.



$s = d^c(x) - 1$ and $t = d^c(y) - 1$.

Let $X = \{x_1, \cdots, x_s\}$ and $Y = \{y_1, \cdots, y_t\}$.

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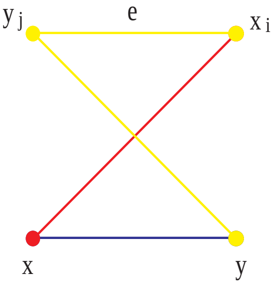
Claim 1. *Suppose* $e = x_i y_j$ $(1 \le i \le s,\ 1 \le j \le t)$ *and* $c(xx_i) \neq c(yy_j)$*. Then* $C(e) \in \{C(xx_i),\ C(yy_j),\ C(xy)\}$*.*



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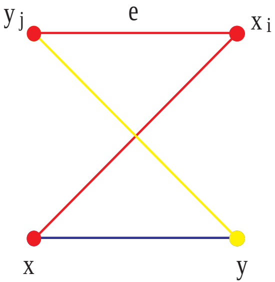

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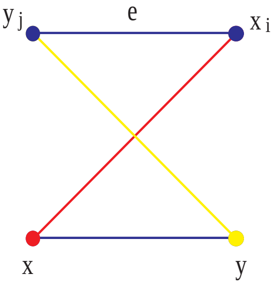
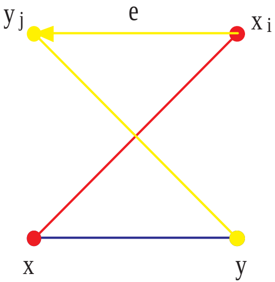


# Heterochromatic $C_4$ in edge-colored triangle-free and bipartite graphs

Given graph $G[X \cup Y]$, let $D_1$ denote the directed bipartite graph obtained by the following operations.
(1) Remove edge $e = x_i y_j$ if $C(e) = C(xy)$ or $C(xx_i) = C(yy_j)$, for $1 \leq i \leq s$ and $1 \leq j \leq t$.
(2) Orient the rest edges by the following rule: for an edge $x_i y_j$, if $C(x_i y_j) = C(yy_j)$, then the orientation of $x_i y_j$ is from $x_i$ to $y_j$; otherwise, by Claim 1, $C(x_i y_j) = C(xx_i)$, then the orientation of $x_i y_j$ is from $y_j$ to $x_i$.



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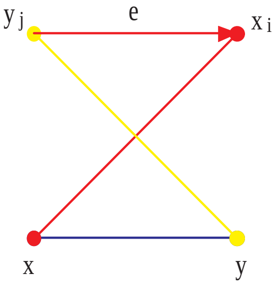

For any vertex $w \in V(D)$, let $N^+_{D_1}(w)$ denote the outneighbors of $w$ in $D_1$ and $d^+_{D_1}(w) = |N^+_D(w)|$.

Claim 2. *If there exists a directed* $\overrightarrow{C_4}$ *in* $D_1$*, then there is an* $HC_4$ *in* $G$*.*

Lemma. *Let* $D$ *be a directed bipartite graph with bipartition* $(A, B)$*. If* $d^+(u) \geq \frac{(3-\sqrt{5})|B|}{2}$ *for* $u \in A$ *and* $d^+(v) \geq \frac{(3-\sqrt{5})|A|}{2}$ *for* $v \in B$*, then there exists a directed* $C_4$ *in* $D$*.*

# Heterochromatic $C_4$ in edge-colored triangle-free and bipartite graphs

$$\begin{aligned}
n \quad & \geq |X| + |Y| + |x| + |y| + |N^c(w) \backslash (Y \cup \{x\})| \\
& > s + t + 2 + d^c(w) - \frac{(3-\sqrt{5})t}{2} - 3 \\
& \geq \frac{3+\sqrt{5}}{2} \times (\frac{(3-\sqrt{5})n}{2} + 1) - \frac{3+\sqrt{5}}{2} \\
& = n.
\end{aligned}$$

Thank you for your attention !