$K_5$-Subdivisions in 5-connected nonplanar graphs

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Joint work with Jie Ma
(Kuartowski): A graph is planar iff it contains no subdivision of $K_5$ or subdivision of $K_{3,3}$. (Denote them as $TK_5$ and $TK_{3,3}$.)

3-Connected nonplanar graphs other than $K_5$ contain $TK_{3,3}$.

Conjecture (Seymour 1977, Kelmans 1979): Every 5-connected nonplanar graph contains a $TK_5$. 

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Background

- Dirac (1964): Any simple graph with $n \geq 5$ vertices and at least $3n - 5$ edges contains a $TK_5$. Erdős and Hajnal (1964) also mentioned this conjecture.


- Conjecture (Hajós): Graphs containing no $TK_5$ are 4-colorable. (Known to Dirac in 1950s)
Background

(Thomassen 1972): Any simple graph with $n \geq 3$ vertices and at least $4n - 10$ edges contains a $TK_5$.

(Thomassen 1972): Let $G$ be a simple graph with $n$ vertices and at least $7n/2 - 7$ edges, and let $v \in V(G)$. Then $G$ contains a $TK_5$ in which $v$ is not a branch vertex.

Theorem (Mader 1998): Any simple graph with $n \geq 3$ vertices and at least $3n - 5$ edges contains a $TK_5$. (Conjectured by Dirac in 1964.)
Our result

- If $G$ is 5-connected and some edge of $G$ is contained in three triangles, then $G$ contains a $TK_5$.

- What about 5-connected nonplanar graphs in which some edge is contained in two triangles? (i.e., Graphs containing $K_4^-$.)

- Theorem (M and X. Yu, 2008): If $G$ a 5-connected nonplanar graph and $K_4^- \subseteq G$, then $G$ contains a $TK_5$.
Future work

- (Kawarabayashi 2001, 2002): 5-Connected graphs without $K_4^-$ contain contractible edges or contractible triangles.

- (Kawarabayashi 2008): Excluding $K_4^-$ and $K_{2,3}$ implies the existence of a nice nonseparating induced cycle.
Let \( x_1, x_2, y_1, y_2 \) be vertices of a \( K_4^- \) in a 5-connected nonplanar graph \( G \) such that \( y_1y_2 \notin E(G) \). Then there is an induced path \( P \) in \( G - x_1x_2 \) between \( x_1 \) and \( x_2 \) such that

- \( \{y_1, y_2\} \not\subseteq V(P) \), and
- \( G - V(P) \) is 2-connected.

So we have two cases.

- \( \{y_1, y_2\} \cap V(P) = \emptyset \) and
- \( y_2 \in V(P), y_1 \notin V(P) \).
Lovász conjecture

Conjecture (Lovász 1975). There is a minimum integer \( c(k) > 0 \) such that for any integer \( k \geq 1 \) and any two vertices \( u \) and \( v \) in a \( c(k) \)-connected graph \( G \), there is a path \( P \) from \( u \) to \( v \) in \( G \) such that \( G - V(P) \) is \( k \)-connected.
\{y_1, y_2\} \cap V(P) = \emptyset

Let \( G \) be a 5-connected nonplanar graph and let \( \{x_1, x_2, y_1, y_2\} \) induce a \( K_4^- \) in \( G \) such that \( y_1y_2 \notin E(G) \). Suppose \( G \) contains an induced path \( X \) from \( x_1 \) to \( x_2 \) such that \( \{y_1, y_2\} \cap V(P) = \emptyset \). Then \( G \) contains a \( TK_5 \) in which \( x_1, x_2, y_1, y_2 \) are branch vertices.
\( \{y_1, y_2\} \cap V(P) = \emptyset \)

(Watkins and Mesner 1967): Let \( R \) be a 2-connected graph and let \( y_1, y_2, v \) be three distinct vertices of \( R \). Then there is no cycle through \( y_1, y_2 \) and \( v \) in \( R \) if, and only if, one of the following holds.
Let $G$ be a connected graph drawn in a closed disc in the plane without edge crossings, and let $a_1, a_2, a_3, a_4, a_5$ be distinct vertices of $G$ on the boundary of the disc, and let $A = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose $G$ is $(5, A)$-connected and $|V(G)| \geq 7$. Let $w \in V(G) - A$ such that the vertices of $G$ cofacial with $w$ induce a cycle $C_w$ in $G - A$.

Then there exist four paths $P_1, \ldots, P_4$ from $w$ to $A$ such that

(i) for $1 \leq i < j \leq 4$, $V(P_i \cap P_j) = \{w\}$, and

(ii) for $1 \leq i \leq 4$, $|V(P_i \cap C_w)| = 1$. 

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Paths in planar graphs
Lemma: Let $G$ be a graph drawn in a closed disc in the plane without edge crossings, and let $a_1, a_2, a_3, a_4, a_5$ be distinct vertices of $G$ on the boundary of the disc in clockwise order, and let $A = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose $G$ is $(5, A)$-connected and $|V(G)| \geq 7$. Then there exists $w \in V(G) - A$, a cycle $C_w$ in $(G - A) - w$, and four paths $P_1, ..., P_4$ from $w$ to $A$ such that

(i) $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C_w)| = 1$ for $1 \leq i \leq 4$, and

(ii) there exist $1 \leq i \neq j \leq 4$ such that $a_1$ is an end of $P_i$ and $a_5$ is an end of $P_j$. 

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Let $G$ be a 5-connected nonplanar graph and let $(G_1, G_2)$ be a 5-separation in $G$. Suppose $|G_2| \geq 7$ and $G_2$ has a planar representation in which the vertices of $V(G_1 \cap G_2)$ are incident with a common face. Then $G$ contains a $TK_5$. 
Theorem (Seymour 1981, Thomassen 1981) Let $G$ be a graph and $s_1, s_2, t_1, t_2$ be distinct vertices of $G$. Then exactly one of the following holds:

- $G$ contains disjoint paths from $s_1$ to $t_1$ and from $s_2$ to $t_2$.
- $(G, s_1, s_2, t_1, t_2)$ is 3-planar.
Let $G$ be a 5-connected nonplanar graph and let \( \{x_1, x_2, y_1, y_2\} \) induce a $K_4^-$ in $G$ such that $y_1y_2 \notin E(G)$. Suppose $G$ contains an induced path $X$ from $x_1$ to $x_2$ such that $y_2 \in P$ and $y_1 \notin P$.

Let $(G, X, x_1, x_2, y_1, y_2)$ be a 6-tuple. Then $G$ contains a $TK_5$, or there exists $z_1 \in V(x_1Xy_2) - \{x_1, y_2\}$ and $z_2 \in V(x_2Xy_2) - \{x_2, y_2\}$ such that $G - (V(X - \{z_1, z_2, y_2\}) \cup E(X))$ has disjoint paths $Z, Y$ from $z_1, y_1$ to $z_2, y_2$ respectively.
Let \((G, X, x_1, x_2, y_1, y_2, z_1, z_2)\) be an 8-tuple. Then \(G\) contains a \(TK_5\), or

1. for any \(i \in \{1, 2\}\), \(H\) contains no path through \(z_i, z_{3-i}, y_1, y_2\) in order, and \(y_1z_i \notin E(G)\), and

2. there exist \(i \in \{1, 2\}\) and independent paths \(A, B, C\) in \(H\) with \(A\) and \(C\) from \(z_i\) to \(y_1\), and \(B\) from \(y_2\) to \(z_{3-i}\).
Let \((G, X, x_1, x_2, y_1, y_2, z_1, z_2)\) be an 8-tuple. Then \(G\) contains a \(TK_5\), or the following holds:

1. there exists disjoint paths \(P, Q\) in \(H\) from \(p, q \in V(B - y_2)\) to \(c \in V(C) - \{y_1, z_1\}\), \(a \in V(A) - \{y_1, z_1\}\), respectively, and internally disjoint from \(A \cup B \cup C\), and

2. \(z_2 x_2 \in E(G)\).
Substructure

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Choice of substructure

- choose \(z_1Xz_2\) maximal.
- choose \(A, B, C\) such that the following are satisfied in the listed order:
  - \(A, B, C\) are induced paths in \(H\),
  - if possible the \((A \cup C)\)-bridge of \(H\) containing \(B\) has attachments on both \(A - \{z_1, y_1\}\) and \(C - \{y_1, z_1\}\),
  - the \((A \cup C)\)-bridge of \(H\) containing \(B\) is maximal, and
  - the union of \(B\) and the \(B\)-bridges of \(H\) not containing \(A \cup C\), denoted by \(B'\), is maximal.
- choose \(P, Q\) such that
  - \(pBz_2\) is maximal and \(qBz_2\) is minimal; and subject to this, \(cCy_1\) is maximal and \(aAy_1\) is minimal.
  - There is no path in \(H\) from \(aAy_1 - a\) to \(z_1Cc - c\) internally disjoint from \(A \cup B \cup C \cup P \cup Q\).
Forcing a 5-separation

- There is a path $R$ in $H$ from $z_1$ to $r \in V(B - y_2)$ internally disjoint from $A \cup B \cup C$.
- There is no path in $H$ from $y_1$ to $B$ internally disjoint from $A \cup B \cup C$.
- There is a 2-cut $\{t_1, t_2\}$ in $H$ separating $\{y_1, z_1\}$ from $\{y_2, z_2\}$, and $\{y_1, y_2, z_1, z_2\} \cap \{t_1, t_2\} = \emptyset$. 
Forcing a 5-separation

\{x_2, y_2, z, t_1, t_2\} is a 5-cut in \(G\), \(G[V(H' \cup zXx_2)]\) is 2-connected and \((5, \{x_2, y_2, z, t_1, t_2\})\)-connected, \(G[V(H' \cup zXx_2)]\) has a plane representation in which \(x_2, y_2, z, t_1, t_2\) occur on a facial cycle in this cyclic order.