K_5-Subdivisions in 5-connected nonplanar graphs

Xingxing Yu

School of Mathematics, Georgia Institute of Technology

Joint work with Jie Ma

Background

- (Kuartowski): A graph is planar iff it contains no subdivision of K_5 or subdivision of $K_{3,3}$. (Denote them as TK_5 and $TK_{3,3}$.)
- 3-Connected nonplanar graphs other than K_5 contain $TK_{3,3}$.
- Conjecture (Seymour 1977, Kelmans 1979): Every 5-connected nonplanar graph contains a TK_5.

Background

- (Dirac 1964): Any simple graph with $n \geq 5$ vertices and at least $3n - 5$ edges contains a TK_5. Erdös and Hajnal (1964) also mentioned this conjecture.
- Kézdy and McGuiness (1991): Seymour's conjecture implies Mader's result.
- Conjecture (Hajós): Graphs containing no TK_5 are 4-colorable. (Known to Dirac in 1950s)

Background

- (Thomassen 1972): Any simple graph with $n \geq 3$ vertices and at least $4n - 10$ edges contains a TK_5.
- (Thomassen 1972): Let G be a simple graph with n vertices and at least $7n/2 - 7$ edges, and let $v \in V(G)$. Then G contains a TK_5 in which v is not a branch vertex.
- Theorem (Mader 1998): Any simple graph with $n \geq 3$ vertices and at least $3n - 5$ edges contains a TK_5. (Conjectured by Dirac in 1964.)

Our result

- If G is 5-connected and some edge of G is contained in three triangles, then G contains a TK_5.
- What about 5-connected nonplanar graphs in which some edge is contained in two triangles? (i.e., Graphs containg K_4^-.)
- Theorem (M and X. Yu, 2008): If G a 5-connected nonplanar graph and $K_4^- \subseteq G$, then G contains a TK_5

Future work

- (Kawarabayashi 2001, 2002): 5-Connected graphs without K_4^- contain contractible edges or contractible triangles.
- (Kawarabayashi 2008): Excluding K_4^- and $K_{2,3}$ implies the existence of a nice nonseparating induced cycle.

Nonseparating paths

Let x_1, x_2, y_1, y_2 be vertices of a K_4^- in a 5-connected nonplanar graph G such that $y_1y_2 \notin E(G)$. Then there is an induced path P in $G - x_1x_2$ between x_1 and x_2 such that

- $\{y_1, y_2\} \not\subseteq V(P)$, and
- $G - V(P)$ is 2-connected.

So we have two cases.

- $\{y_1, y_2\} \cap V(P) = \emptyset$ and
- $y_2 \in V(P), y_1 \notin V(P)$.

Lovász conjecture

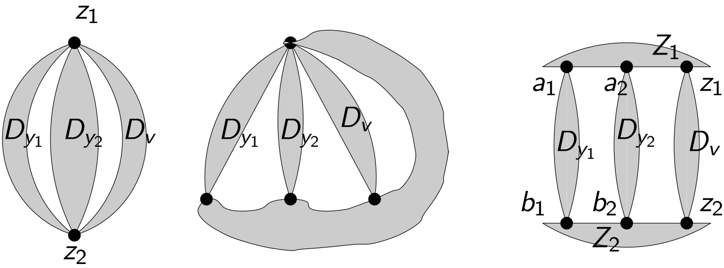
Conjetcure (Lovász 1975). There is a minimum integer $c(k) > 0$ such that for any integer $k \geq 1$ and any two vertices u and v in a $c(k)$-connected graph G, there is a path P from u to v in G such that $G - V(P)$ is k-connected.

$\{y_1, y_2\} \cap V(P) = \emptyset$

Let G be a 5-connected nonplanar graph and let $\{x_1, x_2, y_1, y_2\}$ induce a K_4^- in G such that $y_1 y_2 \notin E(G)$. Suppose G contains an induced path X from x_1 to x_2 such that $\{y_1, y_2\} \cap V(P) = \emptyset$. Then G contains a TK_5 in which x_1, x_2, y_1, y_2 are branch vertices.

$\{y_1, y_2\} \cap V(P) = \emptyset$

(Watkins and Mesner 1967): Let R be a 2-connected graph and let y_1, y_2, v be three distinct vertices of R. Then there is no cycle through y_1, y_2 and v in R if, and only if, one of the following holds.



Paths in planar graphs

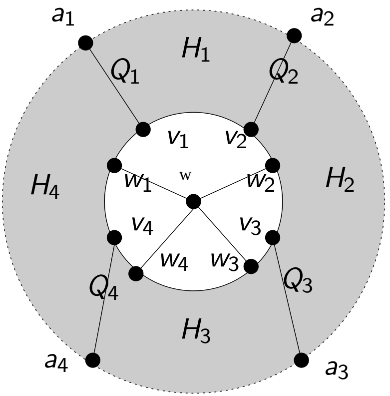
- Let G be a connected graph drawn in a closed disc in the plane without edge crossings, and let a_1, a_2, a_3, a_4, a_5 be distinct vertices of G on the boundary of the disc, and let $A = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose G is $(5, A)$-connected and $|V(G)| \geq 7$. Let $w \in V(G) - A$ such that the vertices of G cofacial with w induce a cycle C_w in $G - A$.

  Then there exist four paths $P_1, \ldots, P_4$ from w to A such that

  (i) for $1 \leq i < j \leq 4$, $V(P_i \cap P_j) = \{w\}$, and

  (ii) for $1 \leq i \leq 4$, $|V(P_i \cap C_w)| = 1$.

Paths in planar graphs



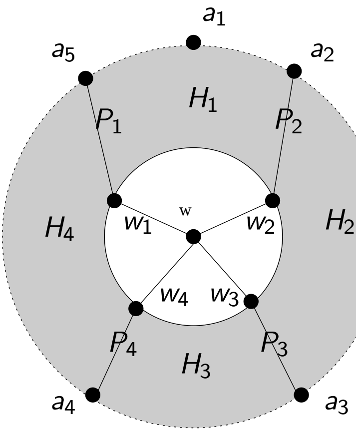
Paths in planar graphs

- Lemma: Let G be a graph drawn in a closed disc in the plane without edge crossings, and let a_1, a_2, a_3, a_4, a_5 be distinct vertices of G on the boundary of the disc in clockwise order, and let $A = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose G is (5, A-connected and $|V(G)| \geq 7$. Then there exists $w \in V(G) - A$, a cycle C_w in $(G - A) - w$, and four paths $P_1, \ldots, P_4$ from w to A such that

  (i) $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C_w)| = 1$ for $1 \leq i \leq 4$, and

  (ii) there exist $1 \leq i \neq j \leq 4$ such that a_1 is an end of P_i and a_5 is an end of P_j.

Paths in planar graphs



Paths in planar graphs

Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|G_2| \geq 7$ and G_2 has a planar representation in which the vertices of $V(G_1 \cap G_2)$ are incident with a common face. Then G contains a TK_5.

Disjoint paths

Theorem (Seymour 1981, Thomassen 1981) Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G. Then exactly one of the following holds:

- G contains disjoint paths from s_1 to t_1 and from s_2 to t_2.
- (G, s_1, s_2, t_1, t_2) is 3-planar.

Substructure

- Let G be a 5-connected nonplanar graph and let $\{x_1, x_2, y_1, y_2\}$ induce a K_4^- in G such that $y_1y_2 \notin E(G)$. Suppose G contains an induced path X from x_1 to x_2 such that $y_2 \in P$ and $y_1 \notin P$.
- Let $(G, X, x_1, x_2, y_1, y_2)$ be a 6-tuple. Then G contains a TK_5, or there exists $z_1 \in V(x_1Xy_2) - \{x_1, y_2\}$ and $z_2 \in V(x_2Xy_2) - \{x_2, y_2\}$ such that $G - (V(X - \{z_1, z_2, y_2\}) \cup E(X))$ has disjoint paths Z, Y from z_1, y_1 to z_2, y_2 respectively.

Substructure

Let $(G, X, x_1, x_2, y_1, y_2, z_1, z_2)$ be an 8-tuple. Then G contains a TK_5, or

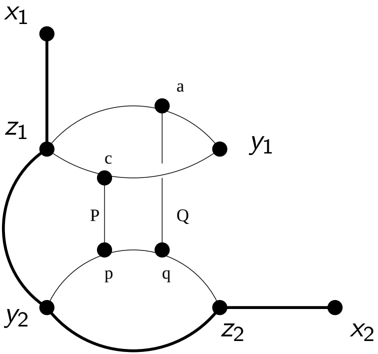
(1) for any $i \in \{1, 2\}$, H contains no path through z_i, z_{3-i}, y_1, y_2 in order, and $y_1 z_i \notin E(G)$, and

(2) there exist $i \in \{1, 2\}$ and independent paths A, B, C in H with A and C from z_i to y_1, and B from y_2 to z_{3-i}.

Substructure

- Let $(G, X, x_1, x_2, y_1, y_2, z_1, z_2)$ be an 8-tuple. Then G contains a TK_5, or the following holds:
  (1) there exists disjoint paths P, Q in H from $p, q \in V(B - y_2)$ to $c \in V(C) - \{y_1, z_1\}, a \in V(A) - \{y_1, z_1\}$, respectively, and internally disjoint from $A \cup B \cup C$, and
  (2) $z_2 x_2 \in E(G)$.

Substructure



Choice of substructure

- choose $z_1 X z_2$ maximal.
- choose A, B, C such that the following are satisfied in the listed order:
  - A, B, C are induced paths in H,
  - if possible the $(A \cup C)$-bridge of H containing B has attachments on both $A - \{z_1, y_1\}$ and $C - \{y_1, z_1\}$,
  - the $(A \cup C)$-bridge of H containing B is maximal, and
  - the union of B and the $B-$bridges of H not containing $A \cup C$, denoted by B', is maximal.
- choose P, Q such that
  - pBz_2 is maximal and qBz_2 is minimal; and subject to this, cCy_1 is maximal and aAy_1 is minimal.
  - There is no path in H from $aAy_1 - a$ to $z_1 Cc - c$ internally disjoint from $A \cup B \cup C \cup P \cup Q$.

Forcing a 5-separation

- There is a path R in H from z_1 to $r \in V(B - y_2)$ internally disjoint from $A \cup B \cup C$.
- There is no path in H from y_1 to B internally disjoint from $A \cup B \cup C$.
- There is a 2-cut $\{t_1, t_2\}$ in H separating $\{y_1, z_1\}$ from $\{y_2, z_2\}$, and $\{y_1, y_2, z_1, z_2\} \cap \{t_1, t_2\} = \emptyset$.

Forcing a 5-separation

$\{x_2, y_2, z, t_1, t_2\}$ is a 5-cut in G, $G[V(H' \cup zXx_2)]$ is 2-connected and $(5, \{x_2, y_2, z, t_1, t_2\})-$connected, $G[V(H' \cup zXx_2)]$ has a plane representation in which x_2, y_2, z, t_1, t_2 occur on a facial cycle in this cyclic order.