Nordhaus-Gaddum-type theorem for diameter of graphs when decomposing into many parts

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Motivation


\[ 2\sqrt{n} \leq \chi(G) + \chi(G') \leq n + 1. \]
Let $K_n$ be the complete graph of order $n$, and $k \geq 2$ a fixed integer. $(G_1, G_2, \ldots, G_k)$ is said to be a $k$-decomposition of $K_n$, if $G_i$ is a spanning subgraph of $K_n$ for each $i = 1, \ldots, k$, $\bigcup_{i=1}^{k} E(G_i) = E(K_n)$ and $E(G_i) \cap E(G_j) = \emptyset$ for any distinct $i, j$.

So, $(G, \overline{G})$ is a 2-decomposition of $G$. 

For a graph parameter \( p \), consider the problem:

\[
? \leq \sum_{i=1}^{k} p(G_i) \leq ?
\]

\( p = \omega, \chi, \chi_l, col(G) \)

\[
? \leq \sum_{i=1}^{k} diam(G_i) \leq ?
\]
1 Main result

Theorem 1.1 Let $K_n$ be the complete graph of order $n$, and $k \geq 2$ a fixed integer. Assume $(G_1, G_2, \ldots, G_k)$ is a $k$-decomposition of $K_n$ such that $G_i$ is connected for each $i = 1, \cdots, k$. Then for any sufficiently large $n$ with contrast to $k$,

$$2k \leq \sum_{i=1}^{k} \text{diam}(G_i) \leq (k - 1)(n - 1) + 2,$$

and the bounds are best possible.
Theorem 1.2 Let $K_n$ be the complete graph of order $n$ and $k \geq 2$ any fixed integer. Then for any sufficiently large $n$ with contrast to $k$, there is a $k$-decomposition $(G_1, G_2, \ldots, G_k)$ of $K_n$ such that $\text{diam}(G_i) = 2$ for each $i = 1, \ldots, k$. 
Proof of Theorem 1.2.

We prove it by the probabilistic argument. Color each edge of $K_n$ by colors $1, 2, \ldots, k$, randomly and independently, with the equal probability $p = \frac{1}{k}$. For each $i, 1 \leq i \leq m$, let $G_i$ denotes the spanning subgraph of $K_n$ with the edge set $E_i$, the set of edges with the color $i$. Hence $(G_1, G_2, \cdots, G_k)$ is a decomposition of $K_n$. Let $A_i$ be the event that $\text{diam}(G_i) \leq 2$. Then $\cap_{i=1}^{k} A_i$ is the event that $\text{diam}(G_i) = 2$ for every $i = 1, 2, \ldots, k$. For two distinct vertices $u, v \in V(G)$, let $B_i(u, v)$ be the event that $d_{G_i}(u, v) > 2$. So,

$$A_i = \cap_{u,v \in V(G)} B_i(u, v).$$

Since $Pr(B_i(u, v)) = (1 - p)(1 - p^2)^{n-2}$, we have
\[ Pr(A_i) = Pr(\bigcap_{u,v \in V(G)} \overline{B_i(u, v)}) \]
\[ = 1 - Pr(\bigcup_{u,v \in V(G)} B_i(u, v)) \]
\[ \geq 1 - \sum_{u,v \in V(G)} Pr(B_i(u, v)) \]
\[ = 1 - \left( \binom{n}{2} \right) (1 - p)(1 - p^2)^{n-2} \]
Since $p = \frac{1}{k} < 1$, $(\frac{n}{2})(1-p)(1-p^2)^{n-2} \to 0$, and $Pr(A_i) \to 1$ when $n \to \infty$. Thus $Pr(A_i \cup A_j) \to 1$, $Pr(A_i \cup A_j \cup A_l) \to 1, \ldots$, $Pr(A_1 \cup A_2 \cup \cdots \cup A_k) \to 1$ when $n \to \infty$. By the principle of inclusion-exclusion,

$$Pr(\cap_{i=1}^{k} A_i) = \sum_{i=1}^{k} Pr(A_i) - \sum_{i<j} Pr(A_i \cup A_j) + \cdots + (-1)^{k-1} Pr(\cup_{i=1}^{k} A_i)$$

$$\to n - \binom{n}{2} + \cdots + (-1)^{k-1}\binom{n}{k} = 1 > 0.$$

It follows that there is a $k$-decomposition $(G_1, G_2, \ldots, G_k)$ of $K_n$ such that $diam(G_i) = 2$ for each $i = 1, \cdots, k$. 
It is interesting to consider the problem that for any fixed integer $k \geq 2$, determine the least $n_k$ such that there exists a $k$-decomposition $(G_1, G_2, \ldots, G_k)$ of $K_{n_k}$ such that $\text{diam}(G_i) = 2$ for each $i = 1, \cdots, k$. Note that $n_2 = 5$, because if $G_1 \cong C_5$, then $G_2 \cong C_5$, and $\text{diam}(G_1) = \text{diam}(G_2) = 2$; but there is the unique decomposition $(G_1, G_2)$ of $K_4$ such that $G_i$ is connected, where $G_i \cong P_4$ for each $i = 1, 2$. 
2 Preparation

Lemma 2.1 Let $G$ be a simple graph with order $n$. If $\delta(G) \geq n/2$, then $diam(G) \leq 2$.

Lemma 2.2 Let $T$ be a tree of order $n$ with $k$ pendent vertices. Then $diam(T) \leq n + 1 - k$.

Lemma 2.3 If $G$ is a simple graph with order $n$, then $diam(G) \leq n + 1 - \Delta(G)$.

Lemma 2.4 (Erdős et al. 1989) For any connected graph $G$ with order $n$,

$$diam(G) \leq \frac{3n}{\delta(G) + 1} - 1.$$
3 Outline of the proof

Fact 1. $\sum_{i=1}^{k} \Delta(G_i) \geq n - 1$.

Fact 2.

$$\Delta(G_i) \leq n - 1 - \sum_{j \neq i} \delta(G_j), \quad \delta(G_i) \geq n - 1 - \sum_{j \neq i} \Delta(G_j).$$

We consider two cases.

Case 1 $\sum_{i=1}^{k} \Delta(G_i) \geq n + 2k - 3$.

By Lemma 2.3, $diam(G_i) \leq n + 1 - \Delta(G_i)$ for each $i = 1, \ldots, k$, and thus

$$\sum_{i=1}^{k} diam(G_i) \leq \sum_{i=1}^{k} (n+1-\Delta(G_i)) \leq k(n+1)-(n+2k-3) = (k-1)(n-1)+2,$$

the result holds.
Case 2 \[ n - 1 \leq \sum_{i=1}^{k} \Delta(G_i) \leq n + 2k - 4. \]

Assume that \( \Delta(G_1) \geq \Delta(G_2) \geq \ldots \geq \Delta(G_k) \).

Claim \( \delta(G_1) \geq \frac{n-2k^2+5k-4}{k} \).

Since
\[
\sum_{i \neq 1} \Delta(G_i) \leq \sum_{i \neq 2} \Delta(G_i) \leq \ldots \leq \sum_{i \neq k} \Delta(G_i),
\]

\[
k \sum_{i \neq 1} \Delta(G_i) \leq (k - 1) \sum_{i = 1}^{k} \Delta(G_i) \leq (k - 1)(n + 2k - 4),
\]

thus
\[
\sum_{i \neq 1} \Delta(G_i) \leq \frac{(k - 1)(n + 2k - 4)}{k}.
\]

By Fact 2, we have
\[
\delta(G_1) \geq n - 1 - \sum_{i \neq 1} \Delta(G_i) \geq n - 1 - \frac{(k - 1)(n + 2k - 4)}{k} = \frac{n - 2k^2 + 5k - 4}{k}.
\]
Subcase 2.1. \(\max\{\delta(G_i) : 2 \leq i \leq k\} \geq 4\).

Without loss of generality, let \(\delta(G_2) \geq 4\). By Lemma 2.4, \(\text{diam}(G_2) \leq \frac{3n}{4+1} = \frac{3n}{5}\). Since \(\delta(G_1) \geq \frac{n-2k^2+5k-4}{k}\), and by Lemma 2.4, we have

\[
\text{diam}(G_1) \leq \frac{3n}{n-2k^2+5k-4} + 1 = \frac{3nk}{n - 2(k^2 - 3k + 2)}.
\]

It is easy to see that if \(n \geq \max\{2(k^2 - 3k + 2), 10k - \frac{5}{2}\}\), then \(\text{diam}(G_1) \leq 4k\) and \(4k + \frac{3n}{5} \leq n + 1\). Therefore,

\[
\sum_{i=1}^{k} \text{diam}(G_i) \leq 4k + \frac{3n}{5} + (k - 2)(n - 1) \leq (k - 1)(n - 1) + 2.
\]
Subcase 2.2. $\max\{\delta(G_i) : 2 \leq i \leq k\} \leq 3$.

Then $\delta(G_i) \leq 3$ for each $i = 2, \ldots, k$, and thus by Fact 2, we have

$$\sum_{j \neq i} \Delta(G_j) \geq n - 1 - \delta(G_i) \geq n - 4.$$

Combining the above with the assumption that $\sum_{i=1}^{k} \Delta(G_i) \leq n + 2k - 4$, it follows that $\Delta(G_i) \leq 2k$. Again by Fact 2,

$$\delta(G_1) \geq n - 1 - \sum_{j \neq 1} \Delta(G_i) \geq n - 1 - 2k(k - 1) \geq \frac{n}{2}$$

for sufficiently large $n$ with respect to $k$. Thus by Lemma 2.1, $diam(G_1) \leq 2$, and hence $\sum_{i=1}^{k} diam(G_i) \leq 2 + (k - 1)(n - 1)$, as we desired.
Thank you!