The Signless Laplacian Spectral Radius of Graphs with Given Degree Sequences

Xiao-Dong ZHANG 张晓东
Shanghai Jiao Tong University
xiaodong@sjtu.edu.cn

Dedicated to professor Tian Feng on the occasion of his 70 birthday
祝田丰教授70岁生日

July 10th- July 11th, GTCA 2010, in Beijing
Outline

- Definition and Notation
Outline

- Definition and Notation
- Prof Tian Feng’s work on spectral graph theory
Outline

- Definition and Notation
- Prof Tian Feng’s work on spectral graph theory
- Problem
Outline

- Definition and Notation
- Prof Tian Feng’s work on spectral graph theory
- Problem
- Main results
Outline

- Definition and Notation
- Prof Tian Feng’s work on spectral graph theory
- Problem
- Main results
- Several Corollaries
Outline

- Definition and Notation
- Prof Tian Feng’s work on spectral graph theory
- Problem
- Main results
- Several Corollaries
- Techniques of the Proofs
Definition and Notation

- $G = (V(G), E(G))$ a simple graph,
- vertex set $V(G) = \{v_1, \cdots, v_n\}$
- edge set $E(G) = \{e_1, \cdots, e_m\}$. 
Definition and Notation

- $G = (V(G), E(G))$ a simple graph,
  - vertex set $V(G) = \{v_1, \cdots, v_n\}$
  - edge set $E(G) = \{e_1, \cdots, e_m\}$.
- $D(G) = \text{diag}(d_1, \cdots, d_n)$: degree diagonal matrix
  - $d_i$: degree of vertex $v_i$ (the number of edges incident to $v_i$).
There are several matrices associated with a graph.
There are several matrices associated with a graph

- \( A(G) = (a_{ij}) \): Adjacency matrix of \( G \), 
  \[ a_{ij} = 1 \text{ if } v_i \sim v_j \text{ and } a_{ij} = 0 \text{ otherwise.} \]
There are several matrices associated with a graph

\[ A(G) = (a_{ij}) : \text{Adjacency matrix of } G, \]

\[ a_{ij} = 1 \text{ if } v_i \sim v_j \text{ and } a_{ij} = 0 \text{ otherwise.} \]

\[ A(G) \] is a nonnegative symmetric \((0, 1)\) matrix with the zeros on the main diagonal.
There are several matrices associated with a graph

$A(G) = (a_{ij})$ : Adjacency matrix of $G$,  
$a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise.

$A(G)$ is a nonnegative symmetric $(0, 1)$ matrix with the zeros on the main diagonal.

The largest eigenvalues of $A(G)$ is denoted by $\rho(G)$, which is called the spectral radius of $G$ or index of $G$. 
**Laplacian Matrix of a graph**

\[
L(G) = D(G) - A(G)
\]
Lapacian Matrix of a graph

\[ L(G) = D(G) - A(G) \]

\[ \lambda_1(G) \geq \lambda_2(G) \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0 \]
Laplacian Matrix of a graph

\[ L(G) = D(G) - A(G) \]

\[ \lambda_1(G) \geq \lambda_2(G) \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0 \]

The smallest eigenvalue \( \lambda_n(G) \) of \( L(G) \) is 0
Laplacian Matrix of a graph

\[ L(G) = D(G) - A(G) \]

\[ \lambda_1(G) \geq \lambda_2(G) \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0 \]

The smallest eigenvalue \( \lambda_n(G) \) of \( L(G) \) is 0

The second smallest eigenvalue \( \alpha(G) \) of \( L(G) \) is positive if and only if \( G \) is connected. \( \alpha(G) \) is called the algebraic connectivity of \( G \).
Laplacian Matrix of a graph

\[ L(G) = D(G) - A(G) \]

- The smallest eigenvalue \( \lambda_n(G) \) of \( L(G) \) is 0.
- The second smallest eigenvalue \( \alpha(G) \) of \( L(G) \) is positive if and only if \( G \) is connected. \( \alpha(G) \) is called the algebraic connectivity of \( G \).
- \( \lambda_{n-1}(G') \leq \kappa(G) \leq \kappa'(G') \).
• Signless Laplacian Matrix of a graph:

\[ Q(G) = D(G) + A(G) \]
• Signless Laplacian Matrix of a graph:

\[ Q(G) = D(G) + A(G) \]

• Signless spectral radius: \( \mu(G) \).

The largest eigenvalue of \( Q(G) \) is called the signless Laplacian spectral radius of \( G \).
Signless Laplacian Matrix of a graph:

\[ Q(G) = D(G) + A(G) \]

Signless spectral radius: \( \mu(G) \).

The largest eigenvalue of \( Q(G) \) is called the *signless Laplacian spectral radius* of \( G \).

The smallest eigenvalue of \( Q(G) \) is 0 if and only if \( G \) is bipartite.
Signless Laplacian Matrix of a graph:

\[ Q(G) = D(G) + A(G) \]

Signless spectral radius: \( \mu(G) \).

The largest eigenvalue of \( Q(G) \) is called the signless Laplacian spectral radius of \( G \).

The smallest eigenvalue of \( Q(G) \) is 0 if and only if \( G \) is bipartite.

\( L(G) \) and \( Q(G) \) are cospectra if and only if \( G \) is bipartite.
Prof. Tian Feng’s work on spectral graph theory

Theorem

(Chang and Tian 2004) Let $\mathcal{U}(2k)$ be the set of all unicyclic graphs on $2k$ vertices with perfect matchings. Denote by $U_{2k}$ the graph on $2k$ vertices obtained from $C_3$ by attaching a pendant edge and $k - 2$ paths of length 2 at one vertex of $C_3$. Then $U_{2k}$ is the only extremal graph with having largest spectral radius of graph in $\mathcal{U}(2k)$.
Prof. Tian Feng’s work on spectral graph theory

**Theorem**

(Liu, Lu and Tian 2004) Let $G$ be a simple graph of order $n$ with edge size $m$. Denote by $\Delta$ and $\delta$ the maximum and minimum degrees, respectively. Then

$$\lambda_1(G) \leq \frac{\Delta + \delta - 1 + \sqrt{\left(\Delta + \delta - 1\right)^2 + 4\left(4m - 2\delta(n - 1)\right)}}{2}.$$
Prof. Tian Feng’s work on spectral graph theory

**Theorem**

(Lu, Liu and Tian 2007) Let $G$ be a simple graph of order $n$ with edge size $m$. Denote by $\Delta$ and $\omega(G)$ the maximum degree and the clique number. Then

$$\omega(G) \geq \frac{2m}{2m - (\mu_1(G) - \Delta)^2}.$$
Degree sequence: \( \pi = (d_0, \cdots, d_{n-1}) \), where \( d_i \) is the degree of vertex \( v_i \) of a graph \( G \) with \( V = \{v_0, \cdots, v_{n-1}\} \).
Degree sequence: $\pi = (d_0, \cdots, d_{n-1})$, where $d_i$ is the degree of vertex $v_i$ of a graph $G$ with $V = \{v_0, \cdots, v_{n-1}\}$.

Graphic: A nonincreasing sequence of nonnegative integers $\pi = (d_0, d_1, \cdots, d_{n-1})$ is called graphic if there exists a simple graph having $\pi$ as its vertex degree sequence.
For example, \( \pi = (3, 3, 2, 2, 2, 2) \) is graphic.

\[
\begin{array}{c}
\text{Graph 1} \\
\text{Graph 2}
\end{array}
\]

\( \pi = (3, 3, 1, 1) \) is not graphic.
For a given graphic degree sequence $\pi$, let

$$G_\pi = \{ G \mid G \text{ is connected with } \pi \text{ as its degree sequence} \}.$$ 

Find the upper (lower) bounds for the signless Laplacian spectral radius of all graphs $G$ in $G_\pi$ and characterize all extremal graphs which attain the upper (lower) bounds.
For example, for a given degree sequence
\( \pi = (4, 4, 3, 3, 3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \), \( T^* \) is the tree of order 17 (see Fig.1). There are a vertex \( v_{01} \) in layer 0; four vertices \( v_{11}, v_{12}, v_{13}, v_{14} \) in layer 1; nine vertices \( v_{21}, v_{22}, \cdots, v_{29} \) in layer 2; three vertices \( v_{31}, v_{32}, v_{33} \) in layer 3. Moreover, \( s_0 = 0, s_1 = d_0 = 4, \)
\( s_2 = d_1 + d_2 + d_3 + d_4 - s_1 = 4 + 3 + 3 + 3 - 4 = 9, \)
\( s_3 = d_5 + \cdots + d_{13} - s_2 = 3, m = s_1 + q = 4 + 2 = 6. \)
Fig. 1. $U_*$. 
Main Results

Theorem

For a given degree sequence of some tree, let

$$\mathcal{T}_\pi = \{ T \mid T \text{ is tree with } \pi \text{ as its degree sequence} \}.$$  

Then $T^*$ a unique tree with largest Laplacian spectral radius in $\mathcal{T}_\pi$. 
Let $\pi = (d_0, d_1, \cdots, d_{n-1})$ be a positive nonincreasing integer sequence with $\sum_{i=0}^{n-1} d_i = 2n$. Then $U^*_\pi$ is the only unicyclic graph having the largest signless Laplacian spectral radius in $G_\pi$. 
Let $\pi = (d_0, \cdots, d_{n-1})$ and $\pi' = (d'_0, \cdots, d'_{n-1})$ be two nonincreasing sequences. If

$$\sum_{i=0}^{k} d_i \leq \sum_{i=0}^{k} d'_i$$

for $k = 0, \cdots, n - 2$ and

$$\sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} d'_i,$$

then the sequence $\pi'$ is said to majorize the sequence $\pi$, which is denoted by $\pi \triangleleft \pi'$. 
Theorem

Let $\pi$ and $\pi'$ be two different degree sequences of tree, or unicyclic graphs with the same order. If $\pi \triangleleft \pi'$, then

$$\mu(U^*_\pi) < \mu(U^*_\pi').$$
Several Corollaries

Corollary

Let \( G \) be any unicyclic graph of order \( n \) with \( s \geq 1 \) leaves; \( G \in \mathcal{U}_{n,s}^{(1)} \). Then \( \mu(G) \leq \mu(U_{\pi_1}^*) \) with equality if and only if \( G \) is \( U_{\pi_1}^* \) with \( \pi_1 = (s + 2, 2, \cdots, 2, 1, \cdots, 1) \), where the frequency of 2 in \( \pi_1 \) is \( n - s - 1 \) and the frequency of 1 is \( s \) (in other words, let \( n - 3 = sq + t \), \( 0 \leq t < s \) and suppose that \( U_{\pi_1}^* \) is obtained from a triangle, \( t \) paths of order \( q + 2 \), and \( s - t \) paths of order \( q + 1 \) by identifying one vertex of triangle and one end of each path of the \( s \) paths).
Corollary

Let $G$ be any unicyclic graph of order $n$ with the maximum degree $\Delta \geq 3$; $G \in \mathcal{U}_{n,\Delta}^{(2)}$. Then $\mu(G) \leq \mu(U_{\pi_2}^*)$ with equality if and only if $G$ is $U_{\pi_2}^*$, where $\pi_2$ is defined as follows: If $\Delta \geq \lceil \frac{n+1}{2} \rceil$, then $\pi_2 = (\Delta, n+1-\Delta, 2, 1, \cdots, 1)$. If $\Delta < \lceil \frac{n+1}{2} \rceil$, then denote by $p = \lceil \log_{(\Delta-1)} \frac{n}{\Delta+1} \rceil - 1$ and $n - (\Delta + 1)(\Delta - 1)^{p-1} = (\Delta - 1)r + q$ for $0 \leq q < \Delta - 1$. If $q = 0$, put $\pi_2 = (\Delta, \cdots, \Delta, 1, \cdots, 1)$, where the frequency of $\Delta$ in $\pi_2$ is $(\Delta + 1)(\Delta - 1)^{p-2} + r$. If $q \geq 1$, put $\pi_2 = (\Delta, \cdots, \Delta, q, 1, \cdots, 1)$, where the frequency of $\Delta$ in $\pi_2$ is $(\Delta + 1)(\Delta - 1)^{p-2} + r$. 
Corollary

Let $G$ be any unicyclic graph of order $n \geq 3$ with the independence number $\alpha$; $G \in U_{n,\alpha}^{(3)}$. Then $\mu(G) \leq \mu(U_{\pi_3}^*)$ with equality if and only if $G$ is $U_{\pi_3}^*$ with $\pi_3 = (\alpha + 1, 2, \cdots, 2, 1, \cdots, 1)$, where the frequency of 2 in $\pi_3$ is $n - \alpha$ and the frequency of 1 is $\alpha - 1$. 
Definition

Let $G = (V, E)$ be a graph with root $v_0$. A well-ordering $\prec$ of the vertices is called a breadth-first search ordering (BFS-ordering for short) if the following holds for all vertices $u, v \in V$:

1. $u \prec v$ implies $h(u) \leq h(v)$;
2. $u \prec v$ implies $d(u) \geq d(v)$;
3. Let $uv \in E(G)$, $xy \in E(G)$, $uy \notin E(G)$, $xv \notin E(G)$ with $h(u) = h(x) = h(v) - 1 = h(y) - 1$. If $u \prec x$, then $v \prec y$.

We call a graph that has a BFS-ordering of its vertices a BFS-graph.
Theorem

If a simple connected graph $G$ has the largest signless Laplacian spectral radius in $G_\pi$, then $G$ has a BFS ordering consistent with the Perron vector $f$ of $G$ in such a way that $u \prec v$ implies $f(u) \geq f(v)$. 
Several Lemmas

**Lemma**

Let $G \in \mathcal{G}_\pi$ with $d_G(u) > d_G(w)$. Let $f$ be the Perron vector of $Q(G)$. If $f(w) \geq f(u)$, then there exists a simple connected graph $G' \in \mathcal{G}_\pi$ such that $\mu(G) < \mu(G')$. 
Lemma

Let $G = (V(G), E(G))$ be a simple connected graph. Assume that $v_1u_1 \in E(G)$, $v_2u_2 \in E(G)$, $v_1v_2 \notin E(G)$ and $u_1u_2 \notin E(G)$. Let $G' = (V(G'), E(G'))$ be a new graph obtained from $G$ by deleting edges $v_1u_1$ and $v_2u_2$ and adding edges $v_1v_2$ and $u_1u_2$. Let $f$ be the Perron vector of $Q(G)$. If $f(v_1) \geq f(u_2)$ and $f(v_2) \geq f(u_1)$, then $\mu(G') \geq \mu(G)$. Moreover, if one of the two inequalities is strict, then $\mu(G') > \mu(G)$. 
Lemma

Let $f$ be the Perron vector of $Q(G')$ with $G \in \mathcal{G}_\pi$. If there exist three vertices $u, v, w$ such that $uv \in E(G)$, $uw \not\in E(G)$, $f(v) < f(w) \leq f(u)$, and $f(u) \geq f(x)$ for all $x \in N(w) = \{x : xw \in E(G)\}$ the neighbor set of $w$. Then there exists a simple connected graph $G' \in \mathcal{G}_\pi$ such that $\mu(G') > \mu(G)$. 


T. Bıyıkoğlu, J. Leydold
Graphs with Given Degree Sequence and Maximal Spectral Radius,

D. Cvetković, P. Rowlinson, S. Simić
Signless Laplacian of finite graphs,

D. Cvetković and S. K. Simić
Towards a spectral theory of graphs based on the signless Laplacian, II,
X. D. Zhang,
The Laplacian spectral radii of trees with degree sequences,

X. D. Zhang,
The signless Laplacian spectral radius of graphs with given degree sequences,
Thank you very much for attention!