The Backup 2-median Problem on Block Graphs

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Outline I

Facility location problem

Given a set of cities, with intercity distances specified, pick p cities for locating warehouses in so as to minimize the total cost.

- Weber problem, Weber (1909).
- The Facility location problem on networks, Hakimi (1964).

Facility location problem

Model the network as a graph $G = (V, E)$ with a length function $l : E \to R_+$ and non-negative vertex weights $w(v)$ for $v \in V$.

Assume the customers coincide with the vertices.

Let distance $d(u, v)$ be the length of the shortest path $P(u, v)$ between vertices u and v.

Denote $d(v, U) = \min_{u \in U} d(u, v)$ for a set $U \subseteq V$.

$A(G)$ is the set of all vertices and all points on edges of networks

The p-center problem

Task

Locate p facilities on edges or vertices of G such that the maximum weighted distance from a vertex (customer) in the network to its nearest center is minimized.

Objective function

Find a set $S \subseteq A(G)$ of p points, such that

$$\max_{v \in V} w(v) d(v, S) = \max_{v \in V} w(v) \min_{u \in S} d(v, u)$$

is minimized.

Kariv and Hakimi (1979) showed that the p-center problem in general graphs, for arbitrary p, is NP-hard.

The p-median problem

Task

Locate p facilities on vertices of G such that the sum of the weighted distance from a vertex (customer) in the network to its nearest median is minimized.

Objective function

Find a set S of p vertices, such that

$$\sum_{v \in V} w(v) d(v, S) = \sum_{v \in V} w(v) \min_{u \in S} d(v, u)$$

is minimized.

The p-median problem

- Mirchandani (1990) showed that the *vertex optimality property* holds for the p-median problem, i.e. there exists an optimal solution which locates all facilities at vertices in V.
- Kariv and Hakimi (1979) showed that the p-median problem in general graphs, for arbitrary p, is NP-hard.

Introduction I

In practice, the uncertainty plays an influential role. Once a set of facilities has been built, one or more of them may become unavailable from time to time. These facility "failure" may result in excessive transportation costs as customers previously served by these facilities must now be served by more distant ones. We should choose the facility locations to minimize the cost while also hedging against failures. Such model is called reliability model.

Introduction II

For the backup p-median problem, the facilities may sometimes fail in a given probability and the clients originally served by these facilities have to be allocated to the functioning facilities.

Wang, Wu and Chao (2009) formally defined the backup 2-median problem and and proposed an $O(n \log n)$ algorithm for this problem.

Introduction III

Task

Minimize the expected value of the sum of distance from all vertices to the functioning facilities.

Assumption

- All facilities do not fail simultaneously.
- The distance between any pair of vertices does not change when a server fails.
- It is not required that the servers are deployed at different vertices.

Problem formulation I

Given an undirected and simple graph $G = (V, E, w, l)$ where $w(v)$ is the non-negative weight of vertex $v \in V$ and $l(e)$ is the positive length of edge $e \in E$.

- The set of p facilities is denoted by a point set $X = \{x_1, \cdots, x_p\}$ where x_i is the location of the ith-facility's location.
- $\rho = \{\rho_1, \ldots, \rho_p\}$ is the failure probability vector where probability ρ_i is only relative to the ith-facility its own.

Problem formulation II

Objective function

For a given point set $X \subseteq A(G)$,

$$C_\rho(X) = \sum_{Y \subseteq X, Y \neq \emptyset} g_\rho(Y) f(Y)$$

where

$$f(Y) = \sum_{v \in V} w(v) d(v, Y), \quad g_\rho(Y) = \prod_{x_i \in Y} (1 - \rho_i) \prod_{x_j \notin Y} \rho_j.$$

Notation and Property I-Property

Lemma

Given an undirected, simple network $G = (V(G), E(G), w, l)$, the vertex optimality property holds for the backup p-median problem on G.

Notation and Property II-Block graph

Definition

- For a connected graph G, a vertex v is a cut vertex if $G - v$ is disconnected.
- A maximal connected induced subgraph without a cut vertex in G is called a block of G.
- A graph G is a block graph if every block is complete.

Note

- Any two blocks in a block graph have at most one vertex in common.
- Let $B_1, B_2, \cdots, B_l$ be all the blocks of G.

Notation and Property III-The skeleton of a block graph

Skeleton $S = (V_S, E_S)$ of block graph G

- $V_S = (\bigcup_{i=1}^{l} B_i) \bigcup V(G)$, $E_S = \{e_{Bv} |$ if v is in block $B\}$;
- $w_S(x) = \begin{cases} w_G(x), & x \in V(G); \\ 0, & x \in V(S) \setminus V(G). \end{cases}$ length$(e_{Bv}) = \frac{1}{2}$.

- If a block graph is connected, then its skeleton is a tree.
- Skeleton S can be constructed in $O(n + m)$ by the depth-first-search method (Aho et al. [1974]).
- $O(V(S)) = O(E(S)) = O(n)$.

For convenience, we refer to the nodes of S as blocks and vertices.

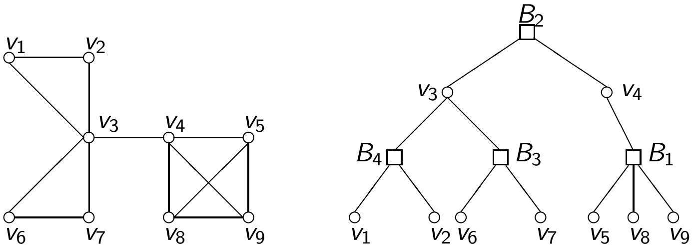
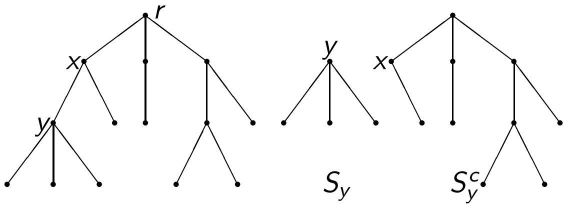


Figure: (a) A block graph G with four blocks; (b) The skeleton of G.

Notation and Property IV-Tree Structure

- Root the skeleton S at root r.
- Vertex x and y are adjacent on S. If x is on the path $P(y, r)$, then x is called the parent of y and y is called a son of x.
- Delete edge (x, y) with $x = par(y)$, skeleton S is decomposed into two subtrees S_y and S_y^c where S_y contains y and the other contains x.



Problem formulation

The 1-median problem on a graph N is just to find a vertex m^* such that the overall sum of the weighted distance of vertices to m^* is minimized.

The objective function for a vertex v and a subgraph N'

$$F(N', v) = \sum_{u \in V(N')} w_N(u) d_N(u, v)$$

By the definition of m^*, $F(N, m^*) = \min_{v \in V(N)} F(N, v)$.

Lemma

If skeleton S's median is a vertex, then this vertex is a median of block graph G.

Lemma

If the median of skeleton S is block B^* and S is rooted at B^*, then son u^* of B^* satisfying $W(S_{u^*}) = \max_{u \in Son(B^*)} W(S_{u_i})$ is the median of block graph G where $Son(B^*)$ is the set of sons of B^*.

$$W(S_u) = \sum_{v \in V(S_u)} w_S(v)$$

Since a tree's median can be found in linear time and $W(S_u)$ for all node $u \in V(S)$ can be computed in $O(n)$ time, thus we have

Theorem

The 1-median problem on a block graph with positive vertex weight and unit edge lengths can be solved in $O(n+m)$ time.

Case for $\rho_1 = \rho_2 = \rho$

Objective function

For a pair of vertices u and v and a given failure probability ρ,

$$\begin{aligned} C_\rho(u,v) &= (1-\rho)\sum_{x\in V(G)} w_G(x)d_G(x,\{u,v\}) + \\ &\quad \rho\Big(\sum_{x\in V(G)} w_G(x)d_G(x,v) + \sum_{x\in V(G)} w_G(x)d_G(x,u)\Big). \end{aligned}$$

An optimal solution $\{m_1, m_2\}$ satisfies

$$C_\rho(m_1,m_2) = \min_{u,v\in V(G)} C_\rho(u,v).$$

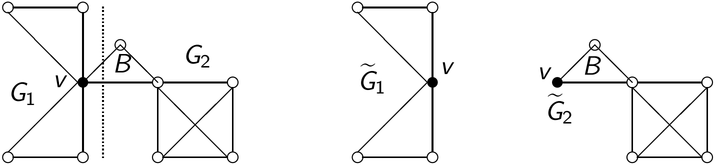
Main idea

In this paper we try to find the relationship of the optimal solutions on a block graph and its skeleton.

By using the algorithm for the backup 2-median problem on trees given by Wang (2009), we can computing an optimal solution on skeleton S. Furthermore we can get an optimal solution on an block graph.

- $\{v, B\}$: the vertex-block pair if vertex v is on block B.
- G_1 and G_2: the subgraphs of G by deleting all edges adjacent to v on B where G_1 contains v.
- $\widetilde{G}_1$ and $\widetilde{G}_2$: the subgraphs gotten by modifying G_1 and G_2. Define $w_{\widetilde{G}_1}(v) = w_G(v) + \delta W(G_2), \quad w_{\widetilde{G}_2}(v) = \delta W(G_1)$. $(\widetilde{G}_1, \widetilde{G}_2)$ is called the graph partition of $\{v, B\}$.
- u_1 and u_2: the median of $\widetilde{G}_1$ and the near median of $\widetilde{G}_2$,

$$F_{\widetilde{G}_1}(u_1) = \min_{u \in V(\widetilde{G}_1)} F_{\widetilde{G}_1}(u), \quad F_{\widetilde{G}_2}(u_2) = \min_{u \in V(\widetilde{G}_2) \setminus \{v\}} F_{\widetilde{G}_2}(u).$$



Rewrite the objective function as the vertex-block pair dependent cost function

$$\begin{aligned} C'_\rho(v,B) &= F_{G_1}(u_1) + \rho F_{G_2}(u_1) + F_{G_2}(u_2) + \rho F_{G_1}(u_2) \\ &= F_{\widetilde{G}_1}(u_1) + F_{\widetilde{G}_2}(u_2) + \rho F_G(v). \end{aligned}$$

The optimal solution $\{m_1, m_2\}$ satisfies

$$C_\rho(m_1, m_2) = \min_{v,B} C'_\rho(v,B).$$

Note

- In order to get the optimal solution, we should find u_1 and u_2 corresponding to all vertex-block pair $\{v, B\}$ and compute the cost function $C'_\rho(v,B)$.
- For each $\{v, B\}$, we compute u_1 and u_2 by using the algorithm for 1-median problem on block graphs respectively on $\widetilde{G}_1$ and $\widetilde{G}_2$. This leads to an $O(n^2 + nm)$ time algorithm.

Improved algorithm

- Root S at median m^*.
- S_y and S_y^c are subtrees gotten by deleting edge (x, y) with $x = par(y)$.
- $X(e)$ and $Y(e)$ are subtrees by modifying S_y^c and S_y with $w_{X(e)}(x) = w_S(x) + \delta W(S_y) w_{Y(e)}(y) = w_S(y) + \delta W(S_y^c)$.
- m_U and m_L are the medians on $X(e)$ and $Y(e)$ respectively. Let v_U and v_L be the vertex medians on $X(e)$ and $Y(e)$ respectively, that is

$$F_{X(e)}(v_U) = \min_{u \in V^G(X(e))} F_{X(e)}(u), F_{Y(e)}(v_L) = \min_{u \in V^G(Y(e))} F_{Y(e)}(u).$$

Main result

Lemma

Let $e = (v, B)$ be an edge in $E(S)$ and $(\widetilde{G}_1, \widetilde{G}_2)$ be the graph partition of $\{v, B\}$.
(1) If $B = par(v)$, then v_L is a median of $\widetilde{G}_1$ with $F\left(\widetilde{G}_1, v_L\right) = F\left(Y(e), v_L\right)$ and v_U is a near median of $\widetilde{G}_2$ with $F\left(\widetilde{G}_2, v_U\right) = F\left(X(e), v_U\right) + \frac{\rho}{2} W(S_v)$;
(2) If $v = par(B)$, then v_L is a near median of $\widetilde{G}_2$ with $F\left(\widetilde{G}_2, v_L\right) = F\left(Y(e), v_L\right) + \frac{\rho}{2}\left(W(G) - W(S_B)\right)$ and v_U is a median of $\widetilde{G}_1$ with $F\left(\widetilde{G}_1, v_U\right) = F\left(X(e), v_U\right)$.

Note

If vertex median v_U and v_L for each edge (v, B) on skeleton can be figured out, then we can compute u_1 and u_2 for $\{v, B\}$ by applying the lemma above.

Suppose $d(m_U)$ and $d(m_L)$ are the degree of m_U and m_L on $X(e)$ and $Y(e)$. Let $X^1(e), X^2(e), \cdots, X^{d(m_U)}(e)$ be the components which are obtained by removing m_U from $X(e)$.
$W(X^i(e)) = \sum_{v \in V(X^i(e))} w_{X(e)}(v)$. Similarly, we can get subtree $Y^i(e)$ and weight $W(Y^i(e))$, $i = 1, 2, \cdots, d(m_L)$.

Lemma

Let $e = (x, y)$ be an edge in $E(S)$. If m_U and m_L are vertices, then $v_U = m_U$ and $v_L = m_L$. Otherwise,
(1) Vertex median v_U of $X(e)$ is the j-th neighbor of m_U on $X(e)$, $j = 1, 2, \cdots, d(m_U)$, satisfying
$W(X^j(e)) = \max_{1 \le i \le d(m_U)} W(X^i(e))$.
(2) Vertex median v_L of $Y(e)$ is the k-th neighbor of m_L on $Y(e)$, $k = 1, 2, \cdots, d(m_L)$, satisfying
$W(Y^k(e)) = \max_{1 \le i \le d(m_L)} W(Y^i(e))$.

We can compute the vertex median v_U and v_L in constant time by applying the lemma above if m_U and m_L have been figured out.

Since m_U and m_L for all edges on skeleton S can be computed in $O(n \log n)$ time and it takes $O(n+m)$ time to construct skeleton S, thus

Theorem

If the failure probability $\rho_1 = \rho_2 = \rho$, the backup 2-median problem on block graphs with positive vertex weights and unit edge lengths can be solved in $O(n \log n + m)$ time

Case for $\rho_1 \neq \rho_2$

- Define $X_i(e)$ and $Y_j(e)$ for edge $e = (x, y)$ where weights of x and y are $w_S(x) + \rho_j W(S_y)$ and $w_S(y) + \rho_i(W(S) - W(S_y))$, respectively, for $i, j \in \{1, 2\}$ and $i \neq j$.
- Define $(\widetilde{G}_{1,i}, \widetilde{G}_{2,j})$ the graph partition of $\{v, B\}$ in which the weight of v are $w_G(v) + \rho_j W(G_2)$ and $\rho_i W(G_1)$. Use the algorithm for the case $\rho_1 = \rho_2 = \rho$ by modifying the failure probability,

Theorem

The backup 2-median problem on a block graph with positive vertex weights and unit edge lengths can be solved in $O(n \log n + m)$ time.

Thank you!