Circular edge cover colorings of cubic graphs

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11,July, 2010

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Edge colorings and Vizing's Theorem

Given a graph $G = (V, E)$, a *k-edge coloring* of G is an assignment of k colors to the edges of G such that the adjacent edges are colored differently. The minimum number k for which a graph G admits a k-edge coloring is the chromatic index $\chi^{'}(G)$ of G.

Theorem(Vizing-1964)
Let G be a graph. Then $\Delta \leq \chi^{'}(G) \leq \Delta + 1$

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Circular chromatic index

Let k, d be positive integers such that $k \geq 2d$. A circular (k, d)-*edge coloring* of G is a mapping $c : E(G) \to \{0, 1, \cdots, k-1\}$ such that for any two adjacent edges x, y, $d \leq |c(x) - c(y)| \leq k - d$.

A circular $(k, 1)$-edge coloring of G is simply a k-edge coloring of G. The *circular chromatic index* $\chi_c'(G)$ is the infimum of the ratios $\frac{k}{d}$ such that G has a circular (k, d)-edge coloring. It is known that the infimum is the minimum for all finite graphs and $\chi'(G) - 1 < \chi_c'(G) \leq \chi'(G)$.

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Partitions

The k-edge coloring is an edge partition $(E_0, E_1, \cdots, E_{k-1})$ of $E(G)$ such that for each j, $0 \le j \le k-1$, E_j induces a matching.

We also can consider a circular (k,d)-edge coloring of a graph G as a (k,d)-*edge partition* of $E(G)$. A (k,d)-edge partition of G is a partition $(E_0, E_1, \cdots, E_{k-1})$ of $E(G)$ such that for each j, $0 \le j \le k-1$, $E_j \cup E_{j+1} \cup \cdots E_{j+d-1}$ is a matching in G, where the addition of indices is taken mod k.

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Circular edge colorings of cubic graphs

The famous theorem of Vizing implies that the chromatic index of every cubic bridgeless graph is 3 or 4.

Theorem(Kaiser et al.-2004)
For every $\epsilon > 0$, there exists g such that every cubic bridgeless graph with girth at least g has circular chromatic index at most $3 + \epsilon$.

Theorem(Afshani et al.-2005)
The circular chromatic index of every cubic bridgeless graph is at most $\frac{11}{3}$.

Theorem(Daniel Král' et al.-2010)
The circular chromatic index of every cubic graph with girth at least 6 is at most $\frac{7}{2}$.

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Edge cover colorings

An *edge cover* of G is a subset S of $E(G)$ that saturates every vertex of G, i.e. every vertex of G is an end vertex of an edge in S. An *edge cover coloring* of G is an edge coloring such that the edges assigned the same color formed an edge cover of G. The *edge cover chromatic index* $\chi'_{ec}(G)$ is the maximum size of a partition of $E(G)$ into edge covers of G.

Theorem(Gupta-1974)
For the simple graph, it holds that $\delta - 1 \leq \chi'_{ec}(G) \leq \delta$.

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Circular edge cover colorings

Let k, d be positive integers such that $k \geq d$. A (k,d)-*edge cover partition* of G is a partition $(E_0, E_1, \cdots, E_{k-1})$ of $E(G)$ such that for each j, $0 \leq j \leq k-1$, and $E_j \cup E_{j+1} \cup \cdots E_{j+d-1}$ forms an edge cover in G, where the addition of indices is taken mod k.

A circular (k,d)-*edge cover coloring* of G is a mapping $c : E(G) \to \{0, 1, \cdots, k-1\}$ such that $(c^{-1}(0), c^{-1}(1), \cdots, c^{-1}(k-1))$ is a (k,d)-edge cover partition. A circular $(k,1)$-edge cover coloring of G is simply a k-edge cover coloring of G. We define the circular edge cover chromatic index as the supremum of the ratios of $\frac{k}{d}$ such that G has a circular (k,d)-edge cover coloring.

Quesetion: "Supremum" can be replaced by "maximum"?

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Proposition

$\chi'_{ec}(G) \leq \chi'_{cec}(G) < \chi'_{ec}(G) + 1$

Proposition

Let G be a graph without isolated vertex. If its circular chromatic index is at most $\frac{k}{d}$, then the circular edge cover chromatic number is at least $\frac{k}{k-(\delta(G)-1)d}$.

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Corollary
For any real $\epsilon > 0$, there exists a positive integer g such that if G is a bridgeless cubic graph and girth at least g, then the circular edge cover chromatic index of G is at least $3 - \epsilon$.

Theorem
The circular edge cover chromatic index of every bridgeless cubic graph is at least $\frac{5}{2}$.

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1 Contracting the cycles into vertices.

2 Decomposing the edges of matchings into some trails.

3 Coloring the edges of matchings by 0 and 1 alternately.

4 Coloring cycles.

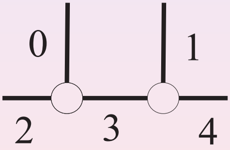

Figure: Coloring odd cycles

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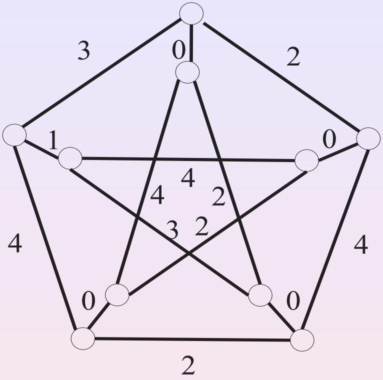

Figure: A circular (5,2)-edge cover coloring of the Peterson graph

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Thanks for your attention!

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