Cycles and paths in graphs

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1. Some definitions and notations

$G = (V(G); E(G))$, $V(G)$ and $E(G)$ are the set of vertices and edges, respectively. $|V(G)| = n$.

A cycle (path) is a **hamiltonian cycle (path)** if it contains all vertices of the graph.

A graph is **hamiltonian** if it contains a hamiltonian cycle.

A graph is **hamiltonian connected** if for any pair of vertices of the graph there is a hamiltonian path connecting them.

A cycle C is **dominating** if $V(G) - C$ is an independent set.

$c(G)$: The circumference of G, which is defined as the length of a longest cycle in G.

$\delta(G) = \min\{d_G(x) : x \in V(G)\}$.

$\sigma_k(G) = \min\{\Sigma_{i=1}^{k} d_G(v_i) :\ \{v_1, v_2, \ldots, v_k\}$ is an independent set of $G\}$.

2. Some basic known results

Theorem 2.1(Dirac, 1952). If $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.

Theorem 2.2(Ore, 1960). If $\sigma_2(G) \geq n$, then G is hamiltonian.

Theorem 2.3(Dirac, 1952). If G is 2-connected, then $c(G) \geq \min\{n, 2\delta(G)\}$.

Theorem 2.4(Bermond, 1972). If G is 2-connected, then $c(G) \geq \min\{n, \sigma_2(G)\}$.

Theorem 2.5(Bauer et al., JCTB 47(1989)). If G is a graph with connectivity $\kappa(G) \geq 2$ such that $\sigma_3(G) \geq n + \kappa(G)$, then G is hamiltonian.

Theorem 2.6(Fournier and Fraisse, JCTB 39(1985)). If G is k-connected $(k \geq 2)$, then $c(G) \geq \min\{n, \frac{2}{k+1}\sigma_{k+1}(G)\}$.

3. Long cycles through a linear forest

A linear forest is a graph F such that each of its components is either a path or an isolated vertex. Set $\mathcal{F}_{m,s} = \{F : F \text{ is a linear forest with } |E(F)| = m \text{ and } |S(F)| = s\}$. $S(F) = \{x \in V(F) : d_F(x) = 0\}$.

A path system is a graph each of whose components is a path of length at least one. A path system is actually a special linear forest.

Remark. If G is k-connected, any vertex set of k vertices is contained in a cycle.

Theorem 3.1(Häggkvist and Thomassen, (1982)). Let G be k-connected $(k \geq 2)$ graph and S be a set of $k-1$ edges in G that induce a path system. Then there exists a cycle in G that includes every edge of S.

Remark. Kawarabayashi recently generalized Theorem 3.1 to a set of k edges if the set is not an edge cut when k is odd.

Theorem 3.2(Egawa et al.,(1991)). Let G be a k-connected $(k \geq 2)$ graph and S be a set of k vertices of G. Then G contains either a cycle of length at least $2\delta(G)$ that includes every vertex of S or a hamiltonian cycle.

Theorem 3.3 (Enomoto, JGT (1984)). Let $m \geq 1$ and let G be a $(m+2)$-connected graph. Then G has a cycle of length $\geq \min\{|V(G)|, \;\; \sigma_2(G) - m\}$ passing through any path of length m.

Theorem 3.4(Hirohata, JGT (1998)). Let $k \geq 3$ and $m \geq 1$ and let G be a $(m+k-1)$-connected graph. Then G has a cycle of length $\geq \min\{|V(G)|, \; \frac{2}{k}\sigma_k(G) - m\}$ passing through any path of length m.

Theorem 3.5(Hu, Tian and Wei, JCTB (2001)). Let $k \geq 3$, $m \geq 0$ and $0 \leq s \leq k-3$. Let G be a $(m+k-1)$-connected graph and let $F \in \mathcal{F}_{m,s}$ be a subgraph of G. Then G has a cycle of length $\geq \min\{|V(G)|, \; \frac{2}{k}\sigma_k(G) - m\}$ passing through $E(F) \cup V(F)$.

4. Dominating cycles in graphs

Theorem 4.1(Bauer et al., JCTB 47(1989)). If G is a graph with connectivity $\kappa(G) \geq 2$ such that $\sigma_3(G) \geq n + \kappa(G)$, then G is hamiltonian.

Theorem 4.2(Bondy (1980)). If G is a graph a 2-connected graph such that $\sigma_3(G) \geq n + 2$, then each longest cycle in G is dominating.

Theorem 4.3(Sun, Tian and Wei, GC 17(2001)). If G is a graph with connectivity $\kappa(G) \geq 3$ such that $\sigma_4(G) \geq n+2\kappa(G)$, then G contains a longest cycle which is dominating.

Theorem 4.4(Lu, Liu, Tian, JGT 49(2005)). If G is a graph with connectivity $\kappa(G) \geq 3$ such that $\sigma_4(G) \geq n + 2\kappa(G)$, then each longest cycle in G is dominating.

Theorem 4.5(Yamashita, DM 308(2008)). If G is a graph with connectivity $\kappa(G) \geq 3$ such that $\sigma_4(G) \geq n + \kappa(G) + 3$, then G contains a longest cycle which is dominating.

Problem 4.6. If G is a graph with connectivity $\kappa(G) \geq 3$ such that $\sigma_4(G) \geq n + \kappa(G) + 3$, is each longest cycle in G dominating?

Problem 4.7. If G is a graph with connectivity $\kappa(G) \geq 4$. Then $c(G) \geq \sigma_4(G) - \kappa(G) - 4$ or each longest cycle in G is dominating?

5. Cycles and paths in claw-free graphs

G is **claw-free**, if there is no induced $K_{1,3}$ in G.

A line graph $L(G)$ of a graph G is a graph in which $V(L(G)) = E(G)$ and where two vertices are adjacent if and only if they are adjacent as edges of G.

Conjecture 5.1 (Matthews & Sumner,JGT, 1983). Every 4-connected claw-free graph is hamiltonian.

Since every line graph is claw-free, the following conjecture proposed in 1986, Thomassen proposed the following conjecture which is a special case of Conjecture 5.1.

Conjecture 5.2(Thomassen,). Every 4-connected line graph is hamiltonian.

An important progress on Conjecture 5.2 is due to Zhan(DM, 91) and independently to Jackson.

Theorem 5.3(Zhan; Jackson). Every 7-connected line graph is hamiltonian.

Theorem 5.4(Ryjácek, JCTB 1997). Every 7-connected claw-free graph is hamiltonian.

Theorem 5.5(Kaiser and Vrána, ENDM 2009). Every 6-connected claw-free graph is hamiltonian.

For Hamilton-connectedness of claw-free graphs, no constant connectivity bound for it was known, until Brandt got the following striking result:

Theorem 5.6 (Brandt, JCTB 1999). Every 9-connected claw-free graph is Hamilton-connected.

Theorem 5.7(Hu, Tian and Wei, JGT 2005). Every 8-connected claw-free graph is Hamilton-connected.

Problem 5.8. Can the connectivity bound for a claw-free graph to be Hamilton-connected be reduced to 7, 6, 5 or even 4?

Panconnectivity and path extendability

For convenience, let $H_u = G[N(u)]$. A vertex u of G is said to be **locally connected** if H_u is connected.

G is called **locally connected** if each vertex of G is locally connected. Generally, G is called **locally k-connected** if for each vertex u, H_u is k-connected.

The distance between two vertices x, y is denoted by $d(x, y)$.

A k-cut is a cut set containing k vertices.

A path with end vertices x and y is called an (x, y)-path. An (x, y)-path is a Hamilton path of G if it contains all the vertices of G.

G is **panconnected** if each pair of distinct vertices x, y are joined by a path of length h for each h, $d(x, y) \leq h \leq n - 1$.

In 1979, Oberly and Sumner proved that *every connected, locally connected, claw-free graph* G *of order* $n \geq 3$ *is hamiltonian.*

Kanetkar and Rao(JGT, 84) improved this result by showing that

Theorem 5.9. Every connected, locally 2-connected, claw-free graph is panconnected.

The following conjecture was proposed by Broersma and Veldman(JGT, 87).

Conjecture 5.10. Let G be a connected, locally connected, claw-free graph of order at least 4. Then G is panconnected if and only if G is 3-connected.

Asratian(JGT, 98) obtained the following weaker result:

Theorem 5.11(Asratian). Let G be a connected, locally connected, claw-free graph of order at least 4. Then G is Hamilton-connected if and only if G is 3-connected.

Theorem 5.12(Sheng, Tian and Wei, DM 1999). Let G be a connected, locally connected, claw-free graph and x, y be two vertices of G. If for any 2-cut S, $S \cap \{x, y\} = \emptyset$, then G contains (x, y)-paths of every length from $d(x, y)$ to $|V(G)| - 1$.

Obviously, Theorem 5.12 implies Conjecture 5.10 and Theorems 5.9 and 5.11.

An (x, y)-path P is **extendable** if there is an (x, y)-path P' such that $V(P') \supset V(P)$ and $|V(P')| = |V(P)|+1$. In the case we say also that P can be extended to P'.

A graph G is said to be **path extendable** if for each pair of vertices x, y and for each nonhamiltonian (x, y)-path P in G, P is extendable.

The following result generalizes all results metioned above:

Theorem 5.13(Sheng et al., DM 2006) Let G be a connected, locally connected, claw-free graph and x, y be any two vertices of G. If for any 2-cut S, $S \cap \{x, y\} = \emptyset$, then each (x, y)-path of G is extendable.

6. Cycles and paths in domination-critical graphs

A subset S of V is called a **dominating set** If for each $x \in V - S$, there exists a vertex $y \in S$ such that $xy \in E(G)$.

Domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G.

A graph G is called k**-domination-critical** (for short, *k-critical*) if $\gamma(G) = k$ and for any $e \notin E$, $\gamma(G+e) = k-1$.

Cycles in 3-Domination-Critical Graphs

Theorem 6.1(Wojcicka, JGT 1991). Every connected 3-critical graph of order at least 7 has a hamiltonian path.

In the same paper, Wojcicka made the followng conjecture:

Wojcicka's Conjecture. Every connected 3-critical graph with minimum degree $\delta \geq 2$ contains a hamiltonian cycle.

Remark. Let G be a connected 3-critical graph. Then $\delta(G) \geq 2$ if and only if G is 2-connected.

In a series of 3 papers, Favaron, Flandrin, Tian, Wei and Zhang proved the above conjecture:

Theorem 6.2. Every connected 3-critical graph with $\delta \geq 2$ contains a hamiltonian cycle.

Paths in 3-Domination-Critical Graphs

A graph G is called **t-tough** if $t\omega(G-S) \leq |S|$ for any cut-set S of G, where $\omega(G-S)$ is the number of components of $G-S$.

The **toughness** $\tau(G)$ is the largest value of t such that G is t-tough.

It is well-known that if G has a hamiltonian cycle then G is 1-tough.

Theorem 6.3(Chen, Tian, Wei(UM 2002)). Let G be a connected 3-critical graph. Then $\tau(G)=1$ if and only if $\kappa(G)=2$ or G belongs to one of three families of graphs with $\delta=3$, where $\kappa(G)$ is the connectivity of G

To investigate the long paths in 3-domination critical graphs, we denote by $p(u,v)$ the length of the longest path connecting u and v. The codiameter of G, denoted by $d^*(G)$, is defined to be $\min\{p(u,v) \mid u,v \in V(G)\}$.

Theorem 6.4(Chen, Tian, Wei(JGT 2002)). Let G be a 3-connected 3-domination critical graph of order n. Then $d^*(G) \geq n-2$ and the lower bound is sharp.

In a series of papers, the following result is proved:

Theorem 6.5. Let G be a connected 3-critical graph. Then G is Hamilton-connected if and only if $\tau(G) > 1$.

Problem 6.6. Let G be a connected 3-critical graph. Under what condition is

(a) G cycle extendable?

(b) panconnected?

(c) path extendable?

Thank you very much!