Entire colouring of plane graphs

Xuding Zhu

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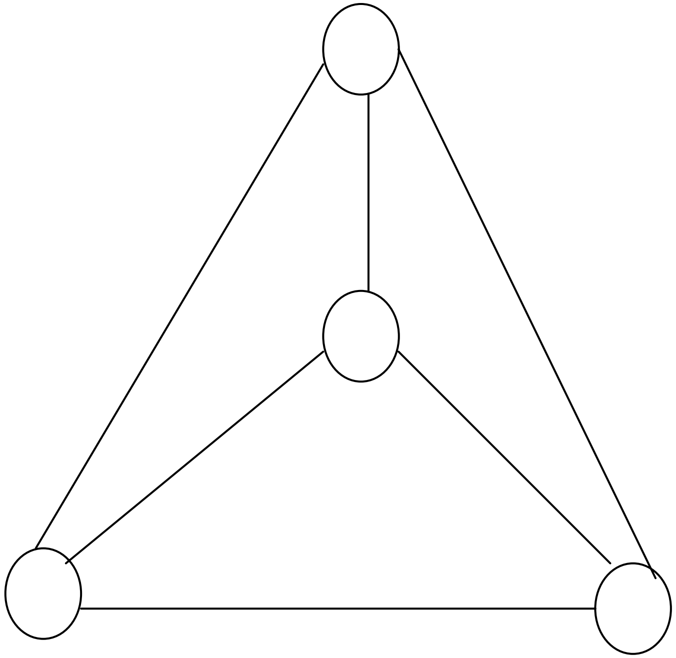
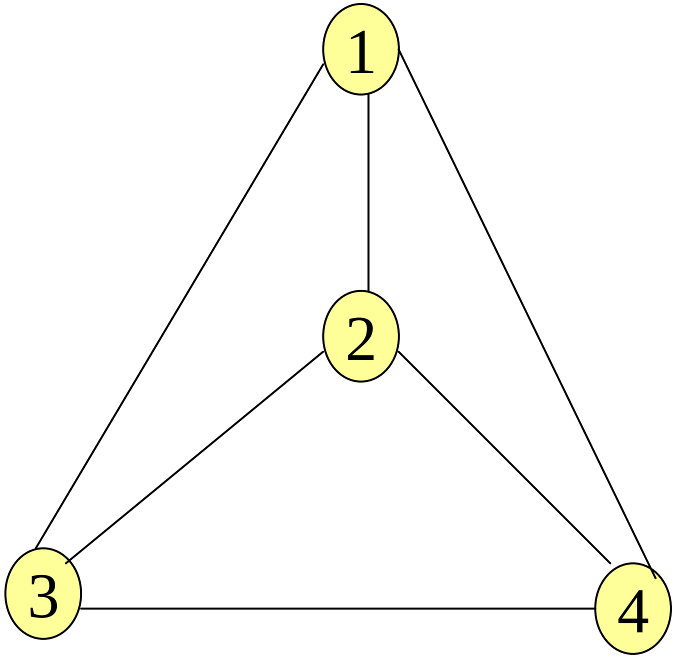
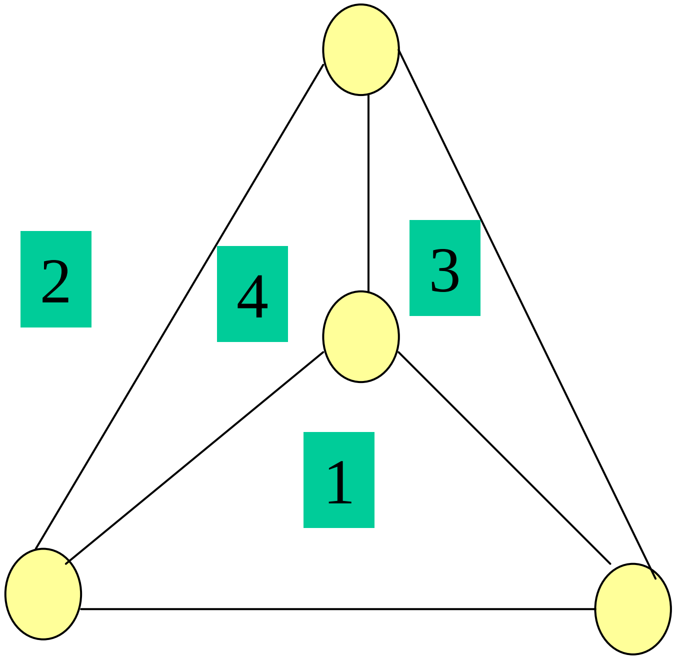
Joint work with Weifan Wang





A proper vertex colouring

Four Colour Theorem: Every planar graph is vertex 4-colourable.



A proper face colouring

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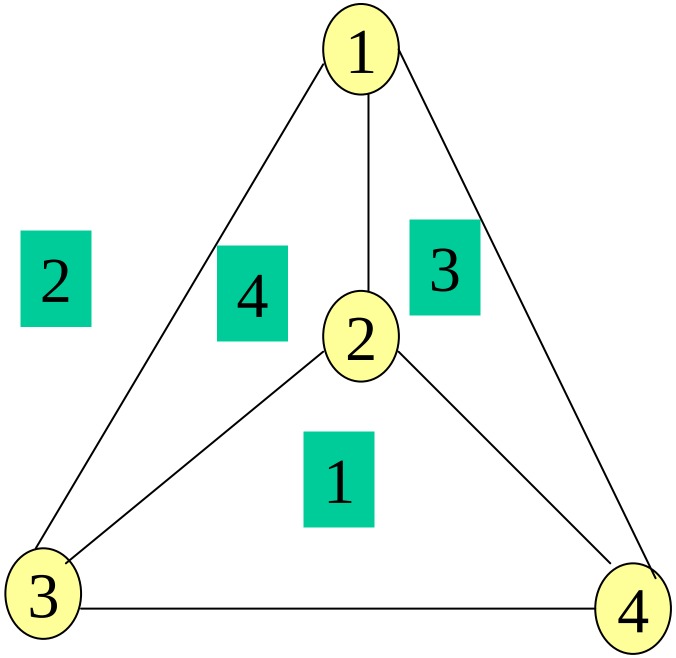
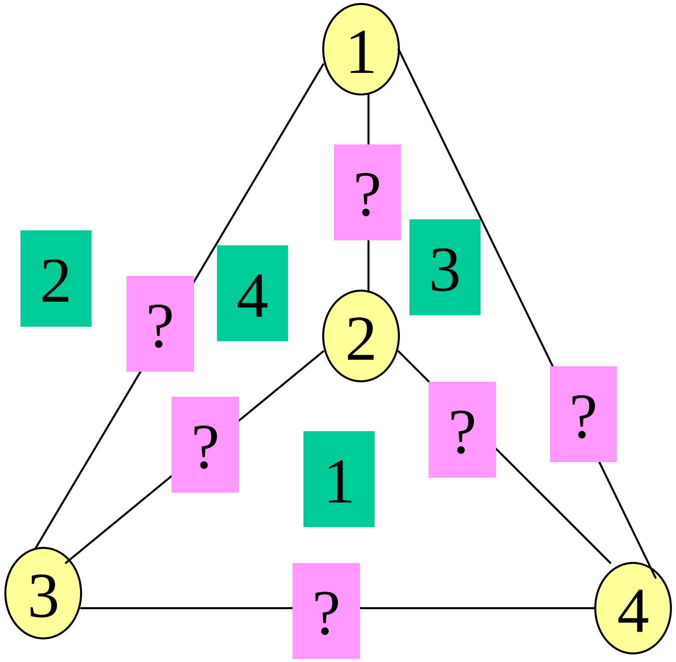
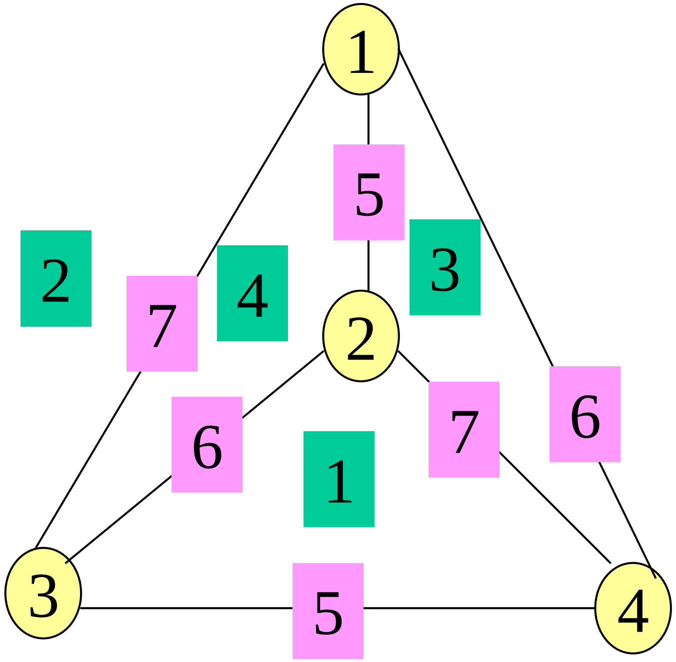


A proper vertex-face colouring

Theorem [Borodin (1995)] Every plane graph is face-vertex 6-colourable



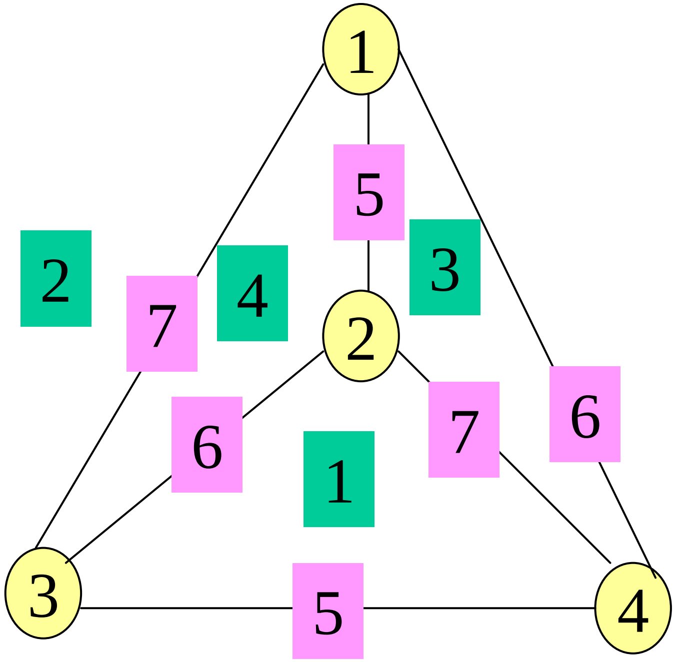
An entire colouring is a proper vertex-face-edge colouring



An entire colouring

Kronk-Mitchem's Conjecture (1972):

Every simple plane graph is entirely $(\Delta + 4)$-colourable.



An entire colouring

This conjecture was confirmed before for $\Delta \geq 6$ and $\Delta = 3$

We prove the whole conjecture.

Theorem [Wang (1999)] Every plane graph is entirely ($\chi'(G)+4$)-colourable

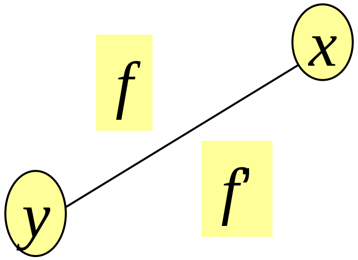
Proof: Assume $\chi'(G)$=k . Colour edges with colours 1,2, …, k.

Uncolour those edges with colour k-1,k.

Colour vertices and faces with colour k-1,k,k+1, …, k+4

Each uncoloured edge e=xy has permissible colours

$$L(e)=\{k-1,k,\ldots,k+4\}-\{c(x),c(y),c(f),c(f')\}$$



|L(e)|=2

The uncoloured edges are L-colourable.

Theorem [Wang (1999)] Every plane graph is entirely ($\chi'(G) + 4$)-colourable

Theorem [Vizing, Sanders-Zhao,Zhang] Every simple plane graph with maximum degree k >= 7 is edge k-colourable

Corollary Kronk-Mitchem's conjecture is true for $\Delta \geq 7$.

$\Delta = 3$

parallel edges allowed

Sanders and Zhao provided a proof for $\Delta = 6$

The proof has an error, but correctable.

Theorem [Wang-Zhu,2010]

If G is a plane graph with $\Delta = 4$ (parallel edges allowed), then G is entirely 8-colourable.

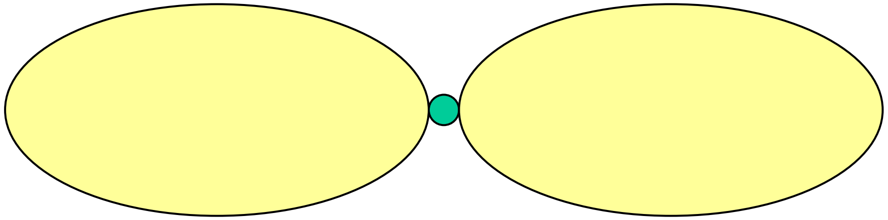
If G is a plane graph with $\Delta = 5$ (parallel edges allowed), then G is entirely 9-colourable.

Theorem [Wang-Zhu,2010]

If G is a plane graph with $\Delta = 4$ (parallel edges allowed), then G is entirely 8-colourable.

Proof: Assume G is a minimum counter-example.

Then G has no cut-vertex



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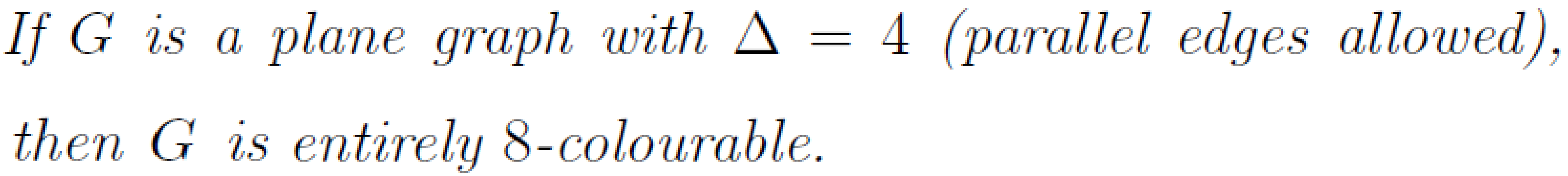

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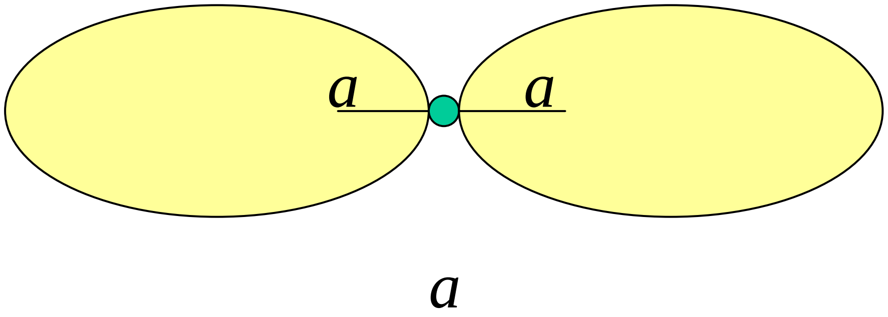

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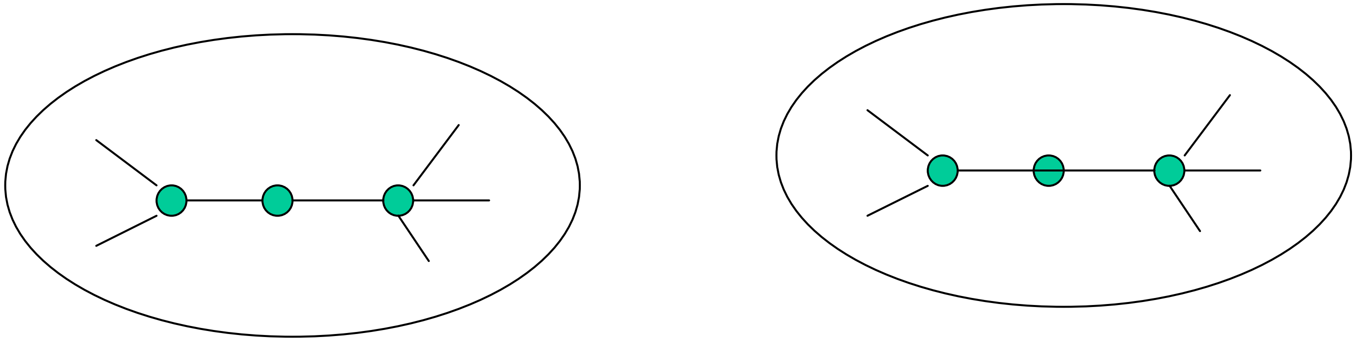
G has no vertex of degree 2.



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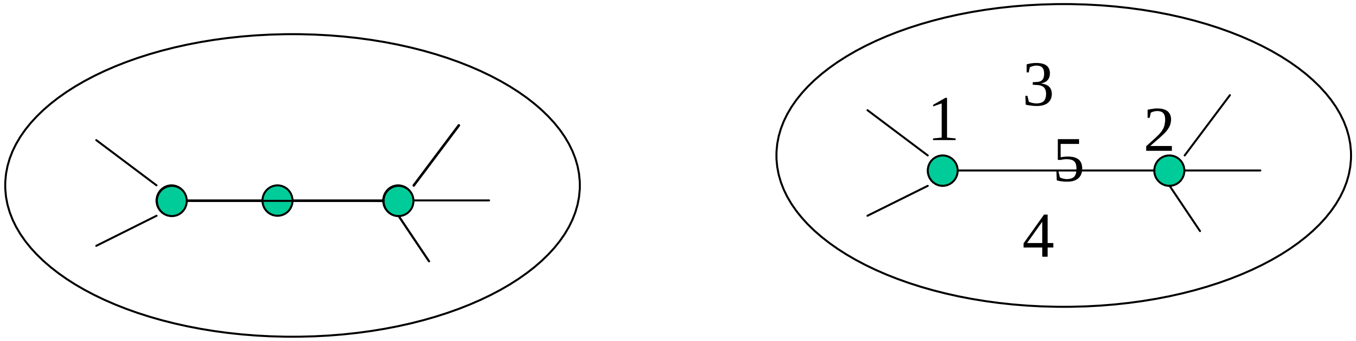

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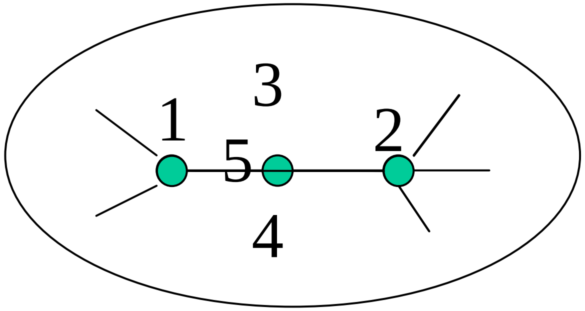

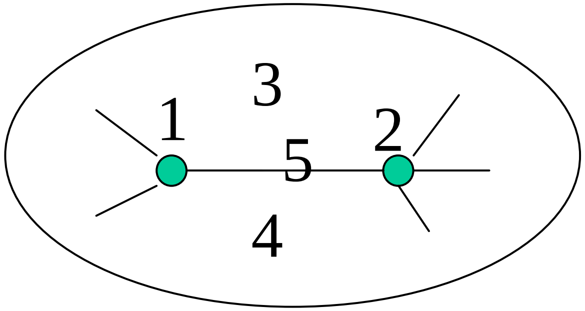

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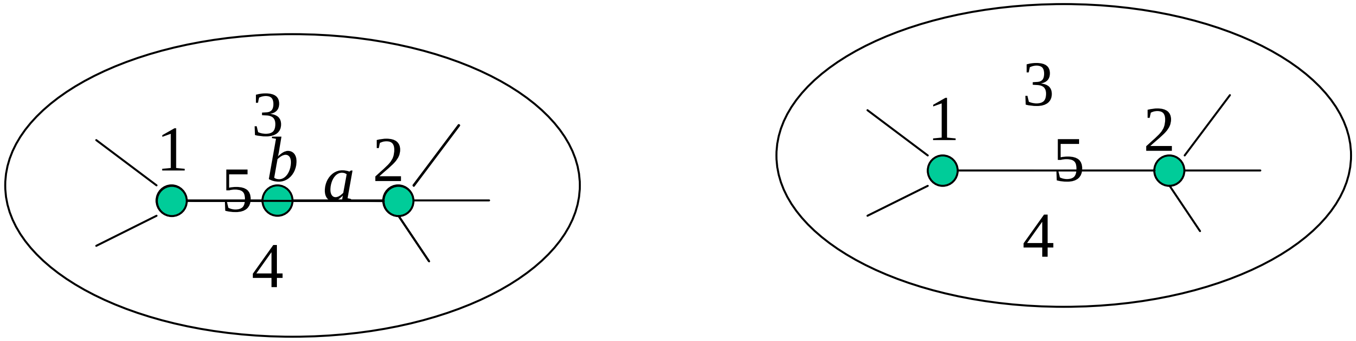

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E can be partitioned into two sets E_1, E_2

$G_i = (V, E_i)$ has maximum degree at most 2.

colour faces of G and edges of G_1 with colours $1, 2, 3, 4.$

colour $V(G) \cup E(G_2)$ with colours $5, 6, 7, 8.$

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By Four Colour Theorem, there is a 4-colouring φ of V(G).

with colours $5, 6, 7, 8.$

Extend φ to $E(G_2)$

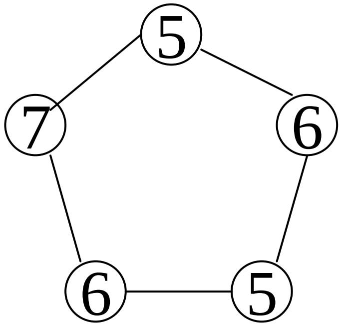
For $e = xy$, $\varphi(e) \in L(e)$ = {5,6,7,8} \ $\{\varphi(x), \varphi(y)\}$

This is possible,

because if C is an odd cycle in G_2, then C has two edges

$e = xy$ and $e' = x'y'$

$\{\varphi(x), \varphi(y)\} \neq \{\varphi(x'), \varphi(y')\}$.



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colour: faces of G and edges of G_1 with colours $1, 2, 3, 4.$

colour $V(G) \cup E(G_2)$ with colours $5, 6, 7, 8.$

Colour the faces properly by colours 1,2,3,4.

For each edge e of G_1 , let L(e) be the two colours from 1,2,3,4 not used by the faces incident to e.

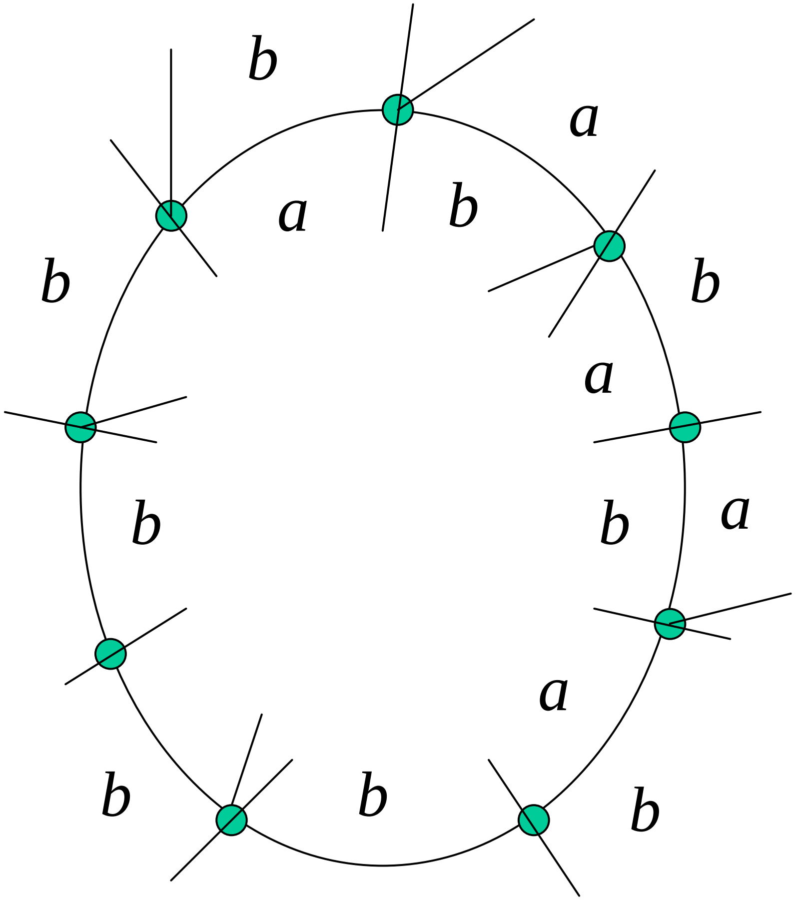
We want to show that the edges of G_1 are L-colourable.

Each component of G_1 is either a path or a cycle.

The only possible problem is odd cycles.

We need to choose the colouring of the faces so that for any odd cycle C in G_1 , L(e) $\neq$ L(e') for some e,e' of C.

colour: faces of G and edges of G_1 with colours $1, 2, 3, 4$.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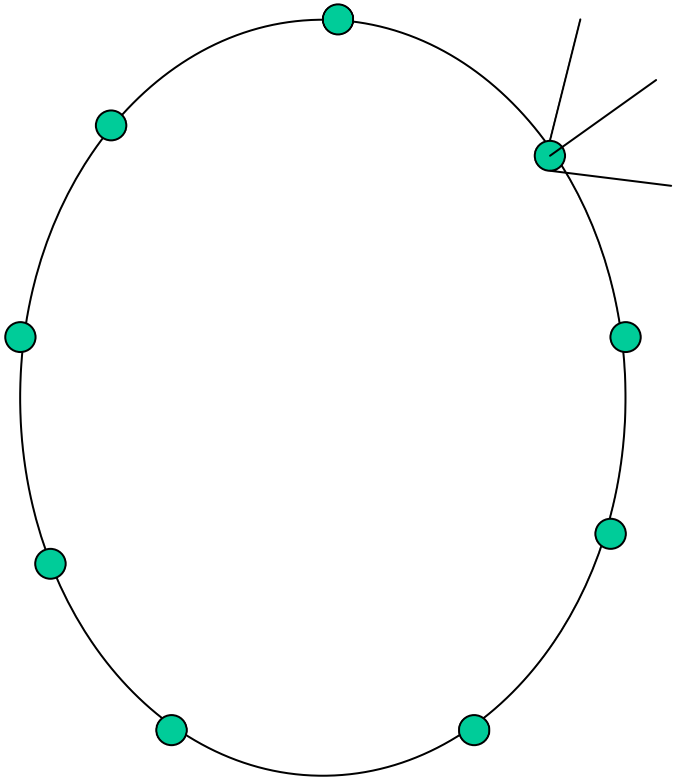

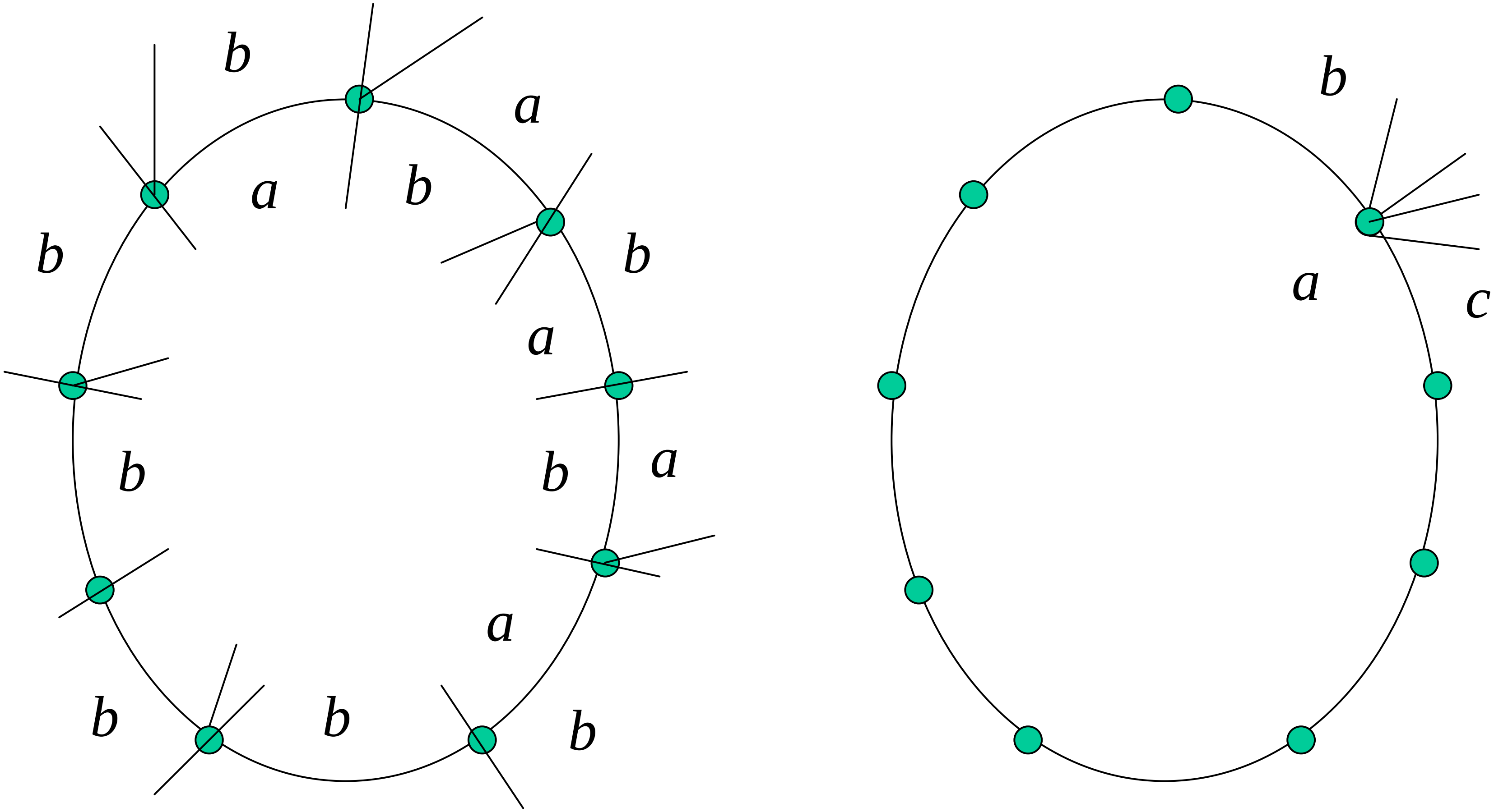
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Theorem [Wang-Zhu,2010]

If G is a plane graph with $\Delta = 5$ *(parallel edges allowed), then G is entirely 9-colourable.*

Proof sketch:

If G has a matching M that covers all the degree 5 vertices, then colour edges in M by colour 9, and colour the other elements of G by colours 1,2,…,8 as before.

This is not always possible.

However, every planar graph G of maximum degree 5 has a matching M for which the following is true:

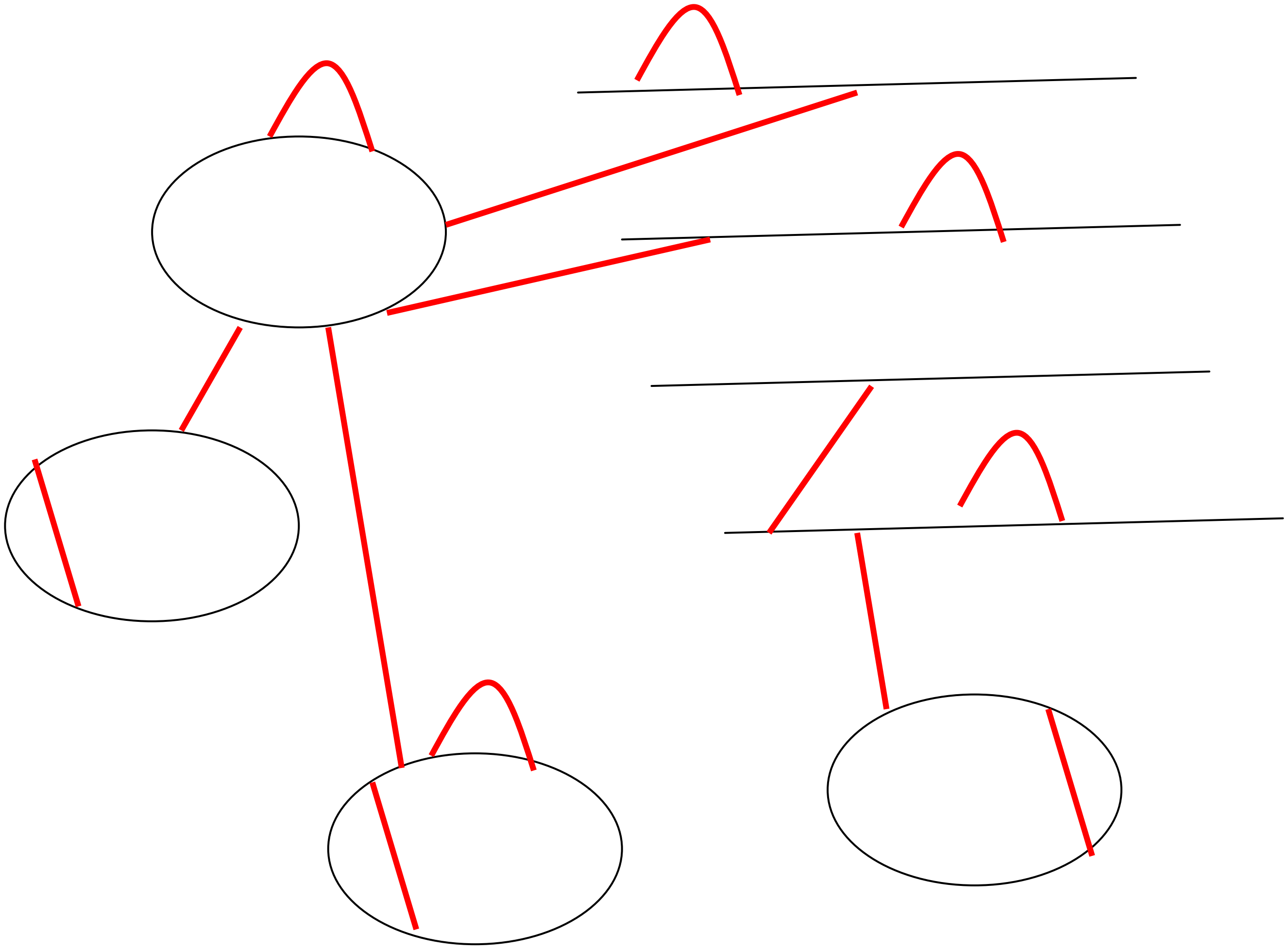
E(G)-M can be partitioned into E_1, E_2

E_1 induces a subgraph of maximum degree 2.

colour faces of G and edges of G_1 with colours $1, 2, 3, 4.$

G_2 = induces a subgraph of maximum degree 2 + some extra edges

Each of the extra edge is incident to a vertex not covered by M.



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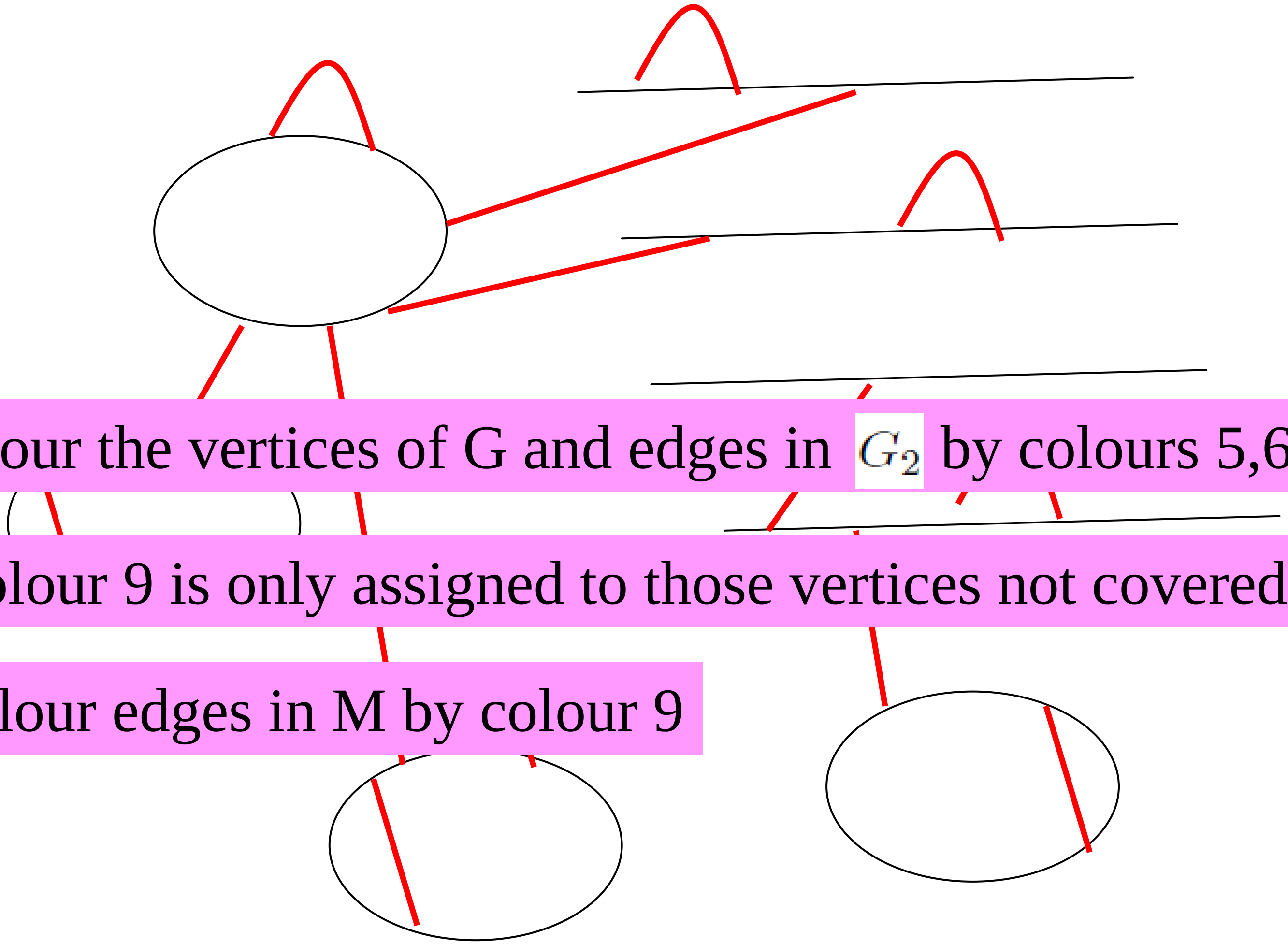


Colour the vertices of G and edges in G_2 by colours 5,6,7,8,9.

Colour 9 is only assigned to those vertices not covered by M

Colour edges in M by colour 9

Welcome to

International Conference on Graph Theory, Combinatorics and Applications

2010.10.29 – 2010.11.2

Zhejiang Normal University

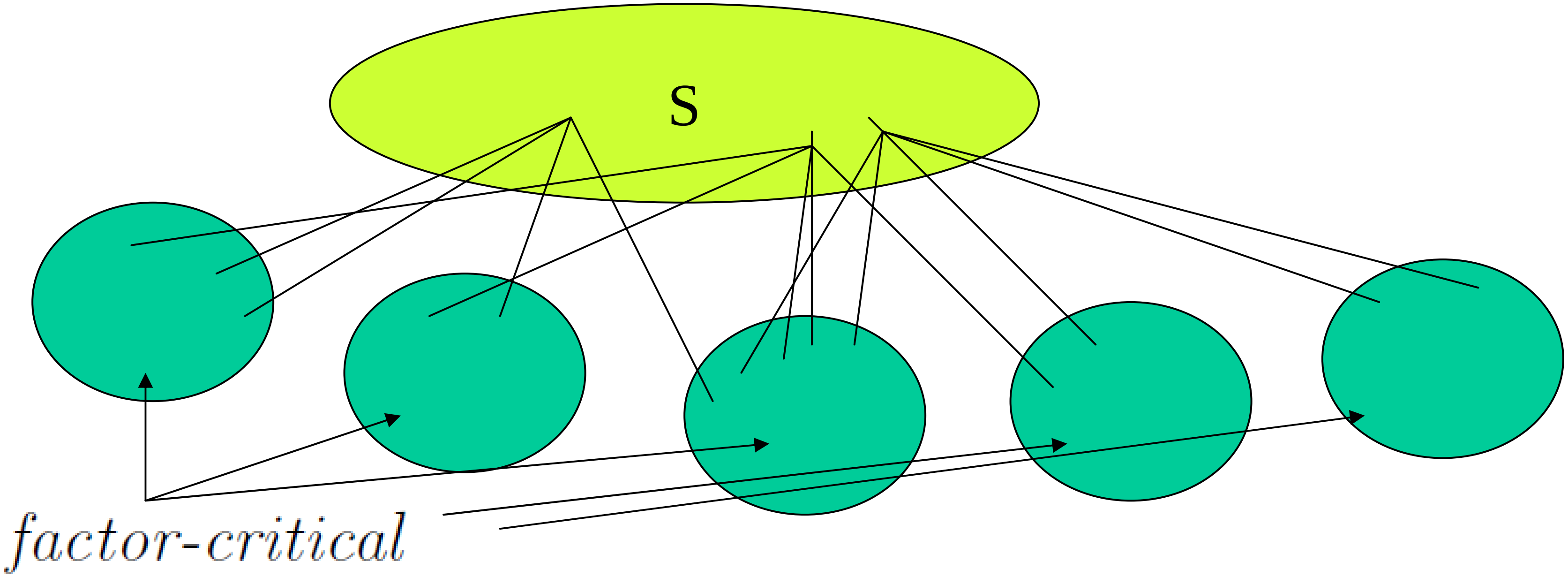
Jinhua, Zhejiang

H is *factor-critical* if $H \neq \emptyset$ and for any vertex v of H, $H - v$ has a perfect matching.

Tutte's Theorem *Every graph G has a vertex set S*

S is matchable to $G - S$

every component of $G - S$ is factor-critical.



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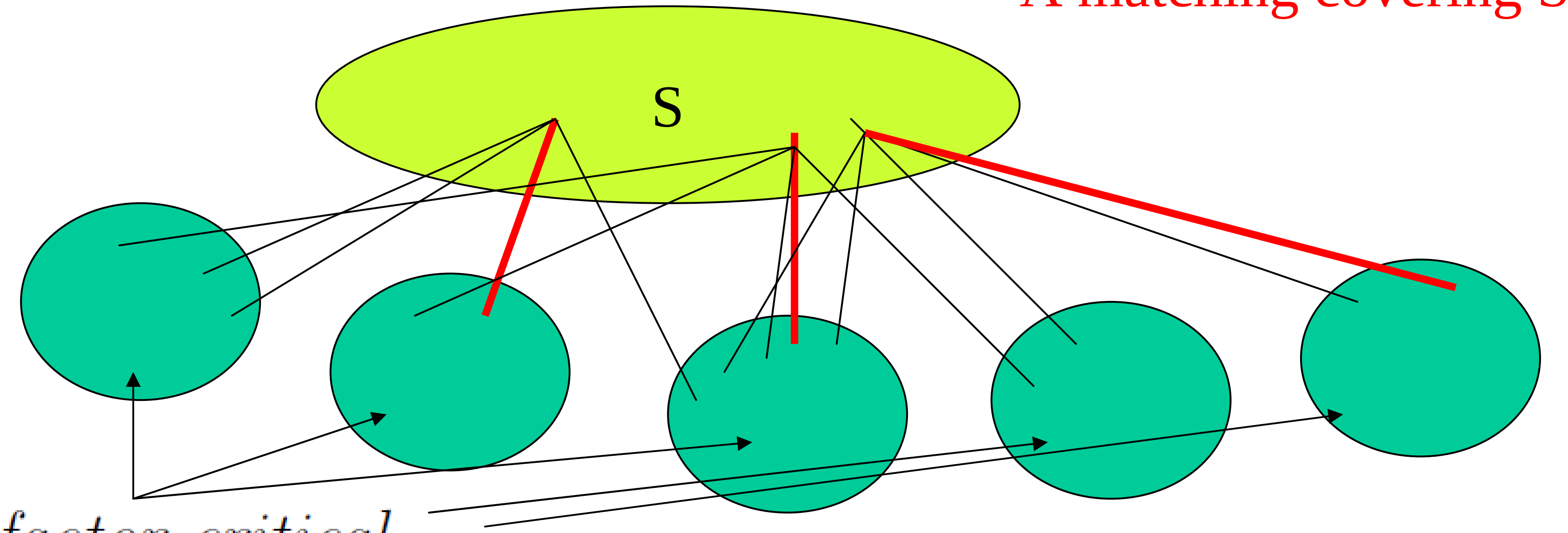

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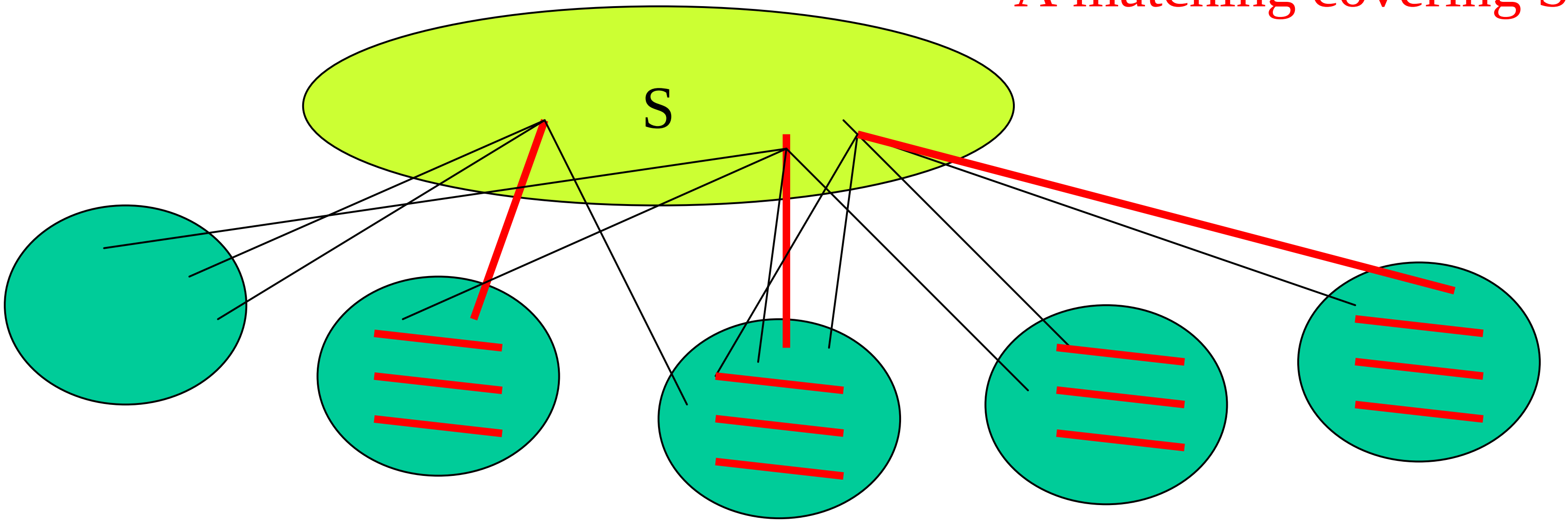
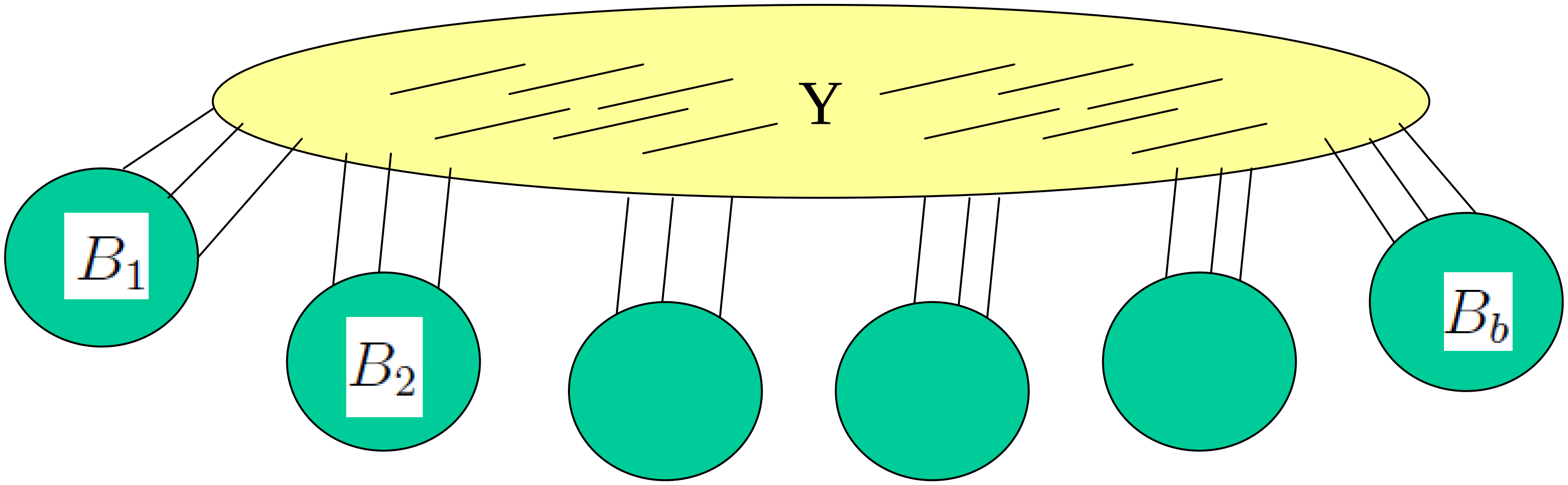
A matching covering S

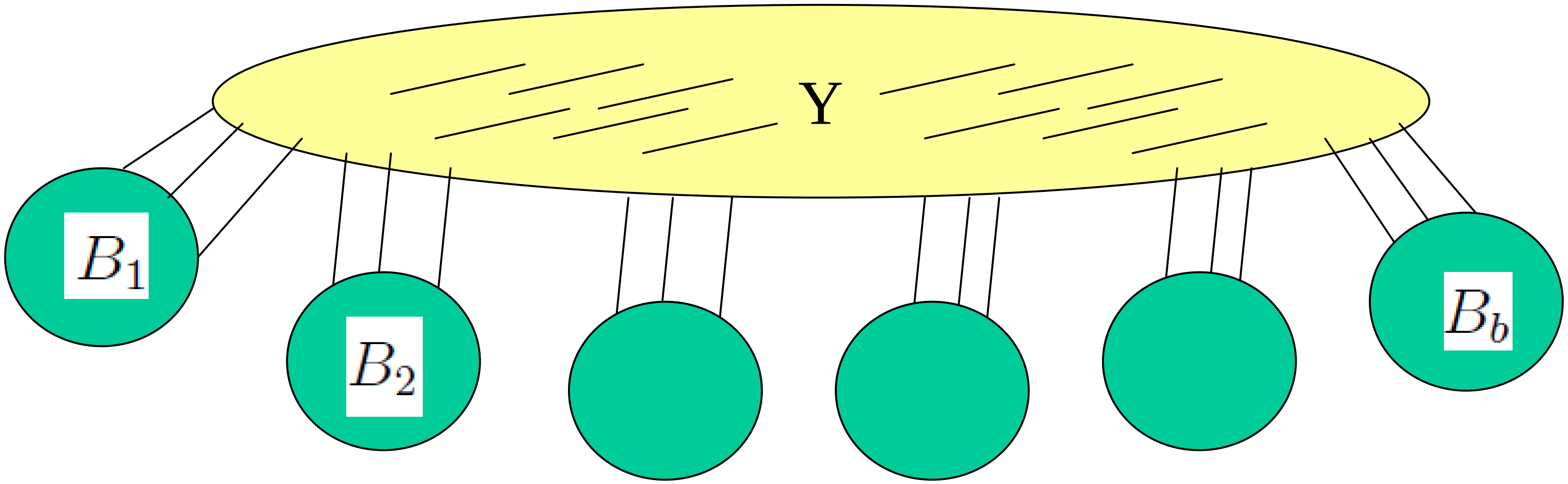




$$\boxed{V_\Delta = \{x \in V(G), d_G(x) = 5\}.}$$

Lemma G has a subset Y of vertices, such that G-Y has components $B_1, B_2, \ldots, B_b$

- *$G[Y]$ has a matching M that covers $Y \cap V_\Delta$.*
- *For $j = 1, 2, \ldots, b$, B_j is factor critical, $B_j \subseteq V_\Delta$ and $|E_G(B_j, Y)| = 3$.*

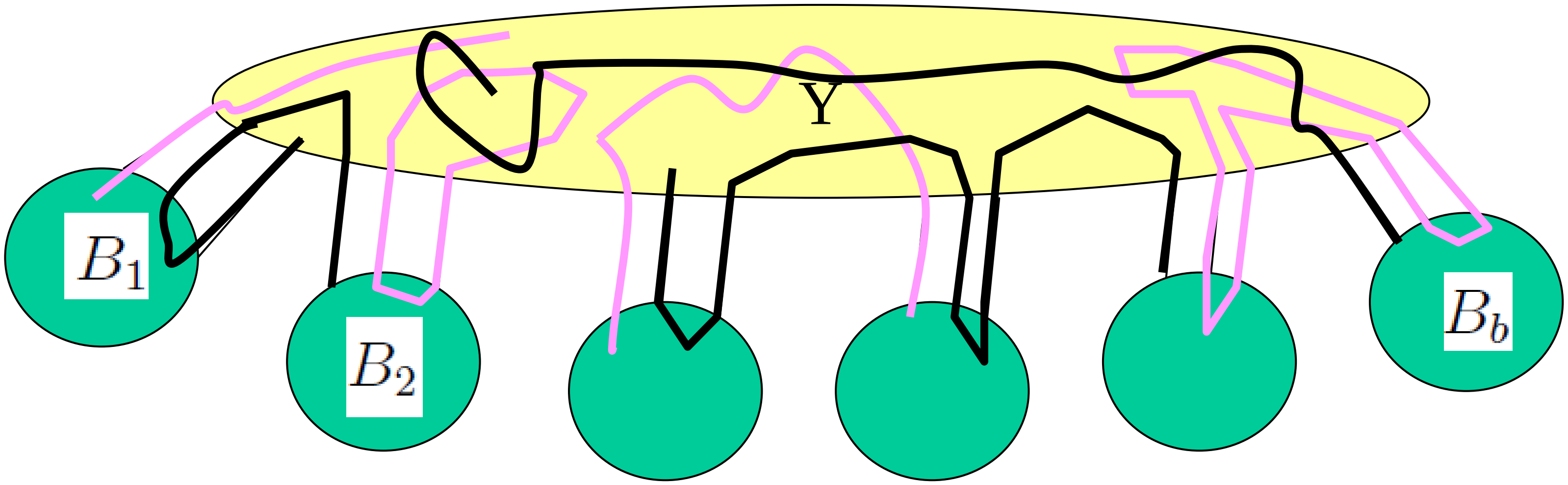


G' be obtained from $G - M$ by contradicting each B_j into a single vertex

G' has maximum degree at most 4

$E(G')$ can be partitioned into E'_1, E'_2

$G'_i = (V(G'), E'_i)$ has maximum degree at most 2



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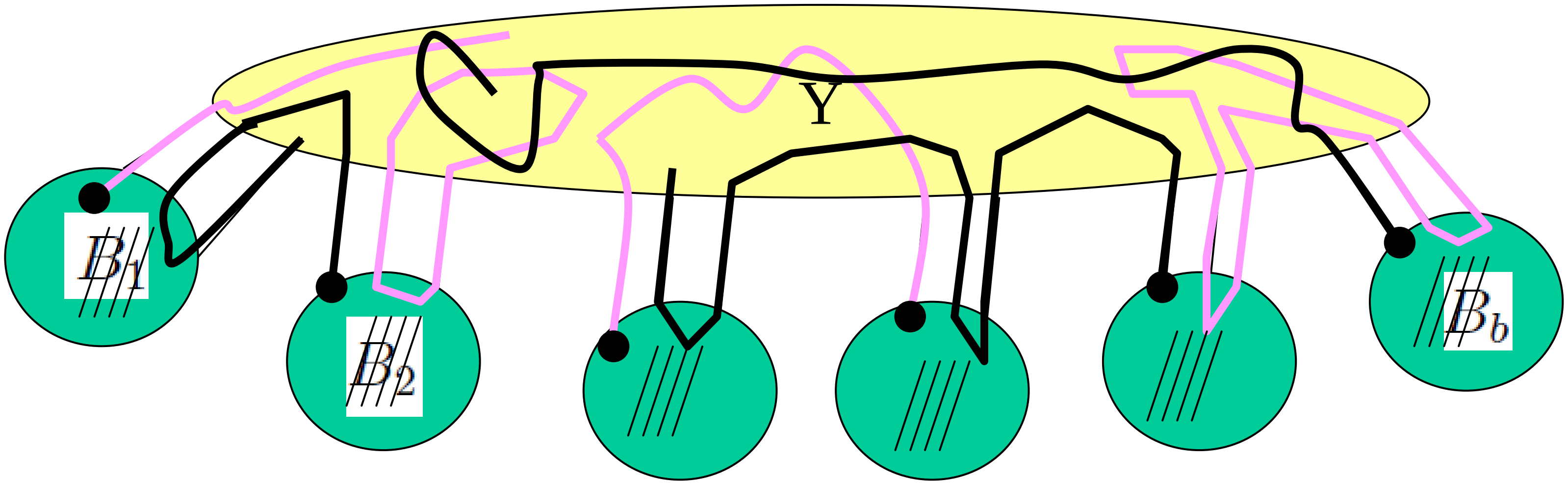
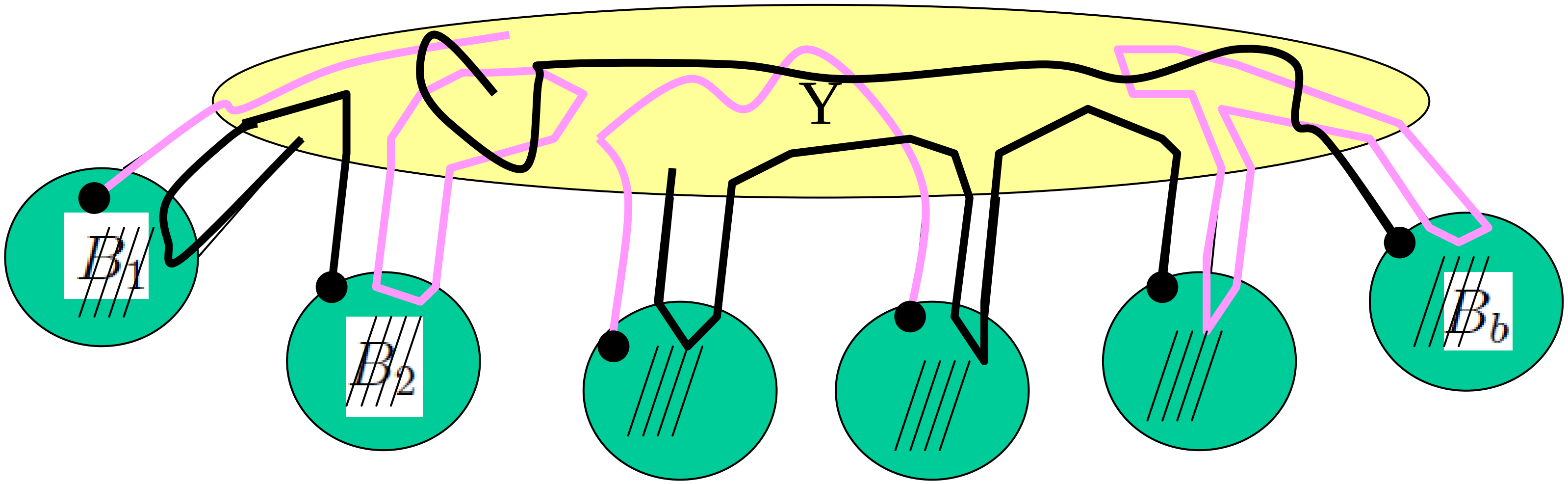


M_j a perfect matching of $B_j - v_{j,1}$

$B'_j = B_j - M_j + v_{j,2}v_{j,3}$ 4-regular.

$G'_i = (V(G'), E'_i)$ has maximum degree at most 2



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$B'_j = B_j - M_j + v_{j,2}v_{j,3}$ 4-regular,

can be partitioned into two 2-factors $E_{j,1}$ and $E_{j,2}$

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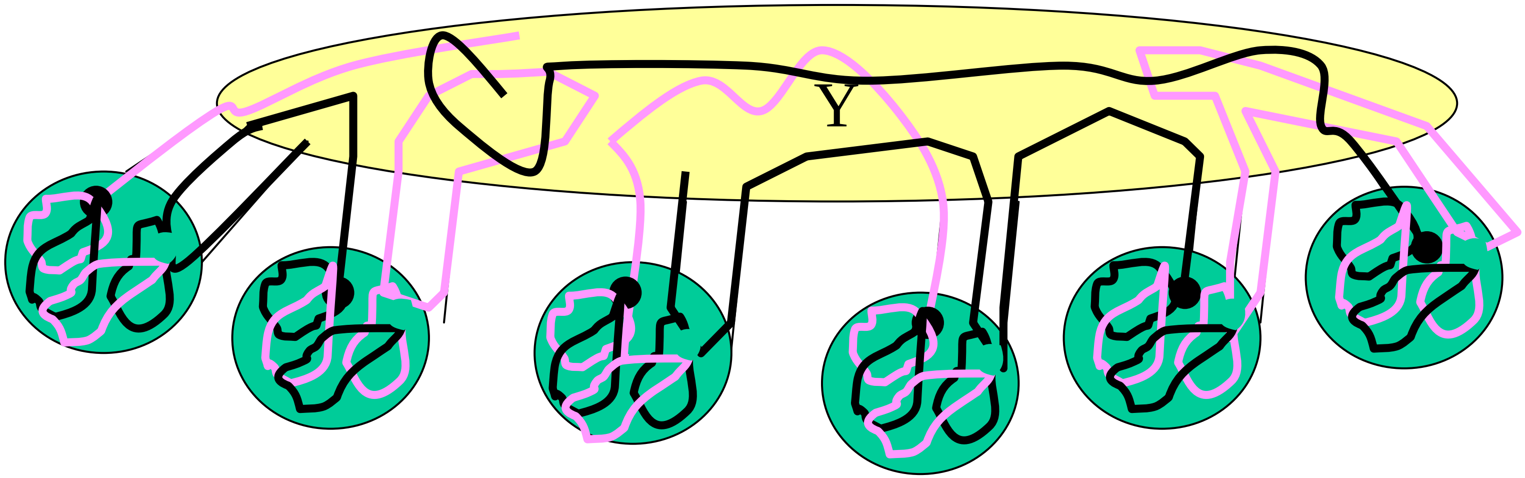

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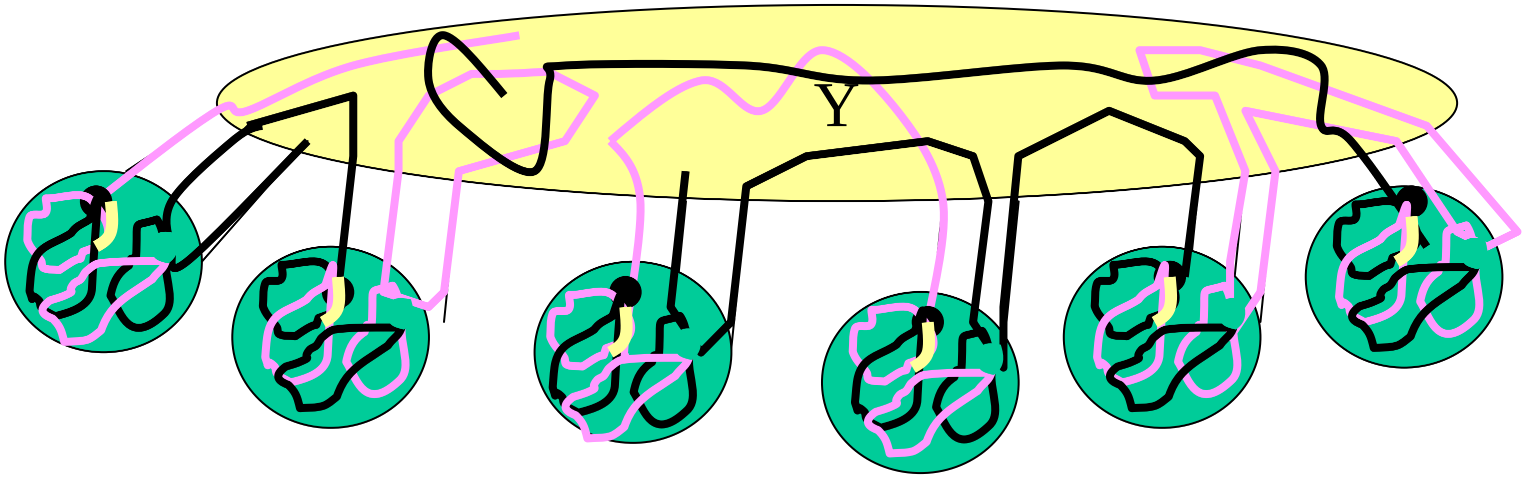


G_1	G_2	Z
the pink graph	the black graph	the set of yellow edges

the faces of G and edges of G_1 can be properly coloured by

$1, 2, 3, 4.$

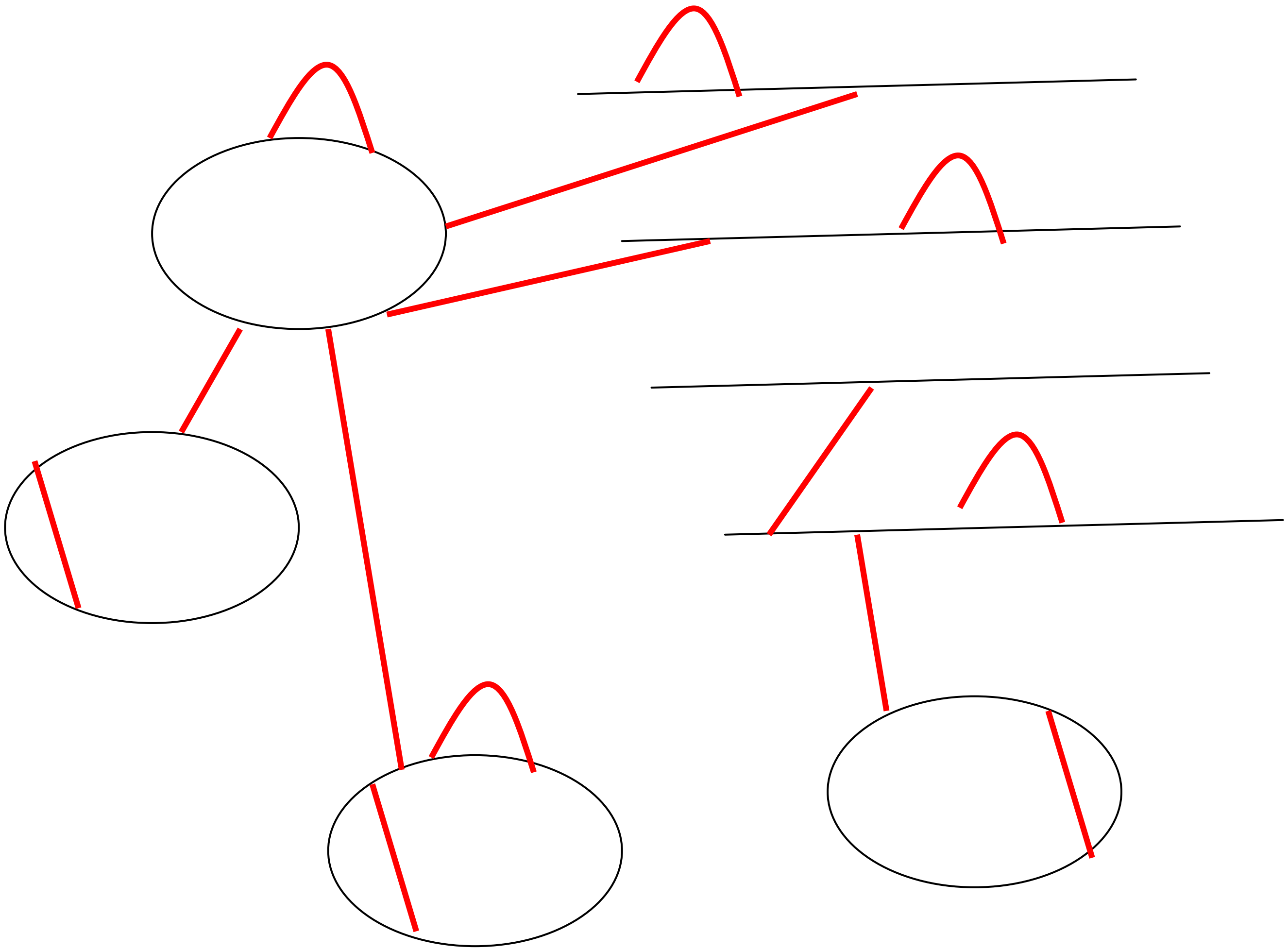
edges in $M \cup \left(\cup_{j=1}^{b} M_j\right)$ and the vertices $v_{j,1}$ colour 9

other vertices of G colours $5, 6, 7, 8$

$G_2 \cup Z$ colours $5, 6, 7, 8$

$G_2 \boxed{\cup Z}$

After the vertices are coloured, each edge has 2 permissible colours, and for each edge e in Z, one end vertex of e has 3 permissible colours.



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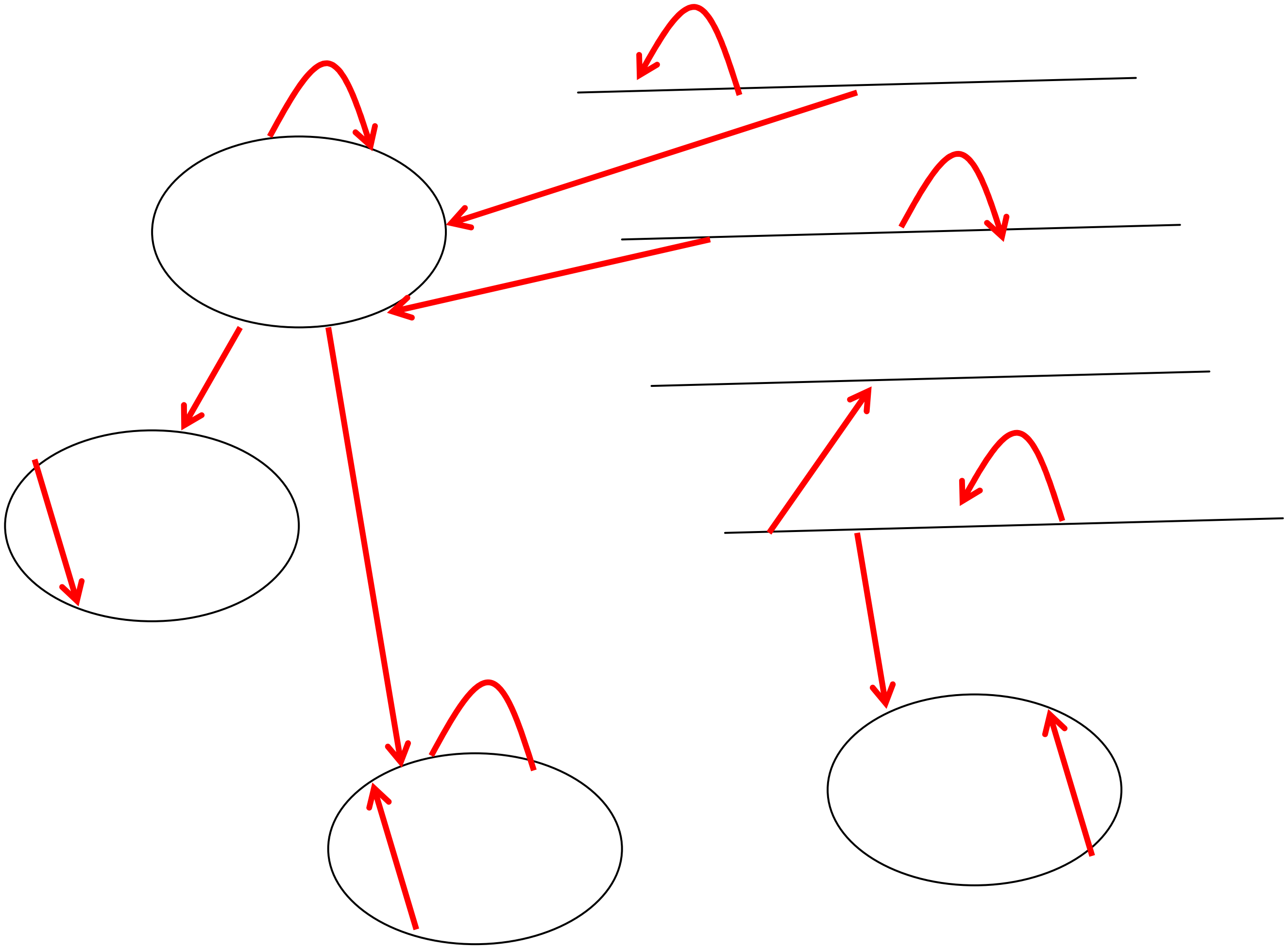

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