Erdős-Ko-Rado Type Theorems

刘九强

西安财经学院

美国东密歇根大学

2010年7月20日

- **Set [n] = { 1, 2, … , n }**

- **A family *F* of subsets of [n] is called *intersecting* if every pair of distinct subsets in *F* have a nonempty intersection.**

Let $L = \{ l_1, l_2, \ldots , l_s \}$.

- A family F of subsets of [n] is called ***L-intersecting*** if

$$|F_1 \cap F_2| \in L$$

for every pair F_1, F_2 from F.

- A family F is ***k-uniform*** if it is a collection of k-subsets of [n].

Erdős-Ko-Rado Theorem (1961)

Let $n \geqslant 2k$ and let F be a k-uniform intersecting family of subsets of $[n]$. Then

$$|F| \leqslant C(n-1, k-1)$$

with equality only when F is a star.

Ray-Chaudhuri - Wilson Theorem

Let $L = \{ l_1, l_2, \ldots, l_s \}$ be a set of s nonnegative integers. If F is a *L-intersecting* family of k-subsets of [n], then

$$|F| \leqslant C(n, s) .$$

- **Example:**

Let *F* be the set of s-subsets of [n]

$$L = \{\, 0, 1, \ldots, s-1 \,\}$$

***Then F* is a *L-intersecting* family of subsets of [n] with**

$$|F| = C(n, s)$$

Frankl and Wilson Theorem (1981)

Let $L = \{ l_1, l_2, \ldots, l_s \}$ be a set of s nonnegative integers. If F is a *L-intersecting* family of subsets of [n], then

$|F| \leqslant$

$C(n, s) + C(n, s - 1) + \ldots + C(n, 0)$.

- **Example:**

Let F be the set of subsets of [n] of sizes ⩽ s and $L = \{ 0, 1, \ldots, s-1 \}$

Then **F is a** ***L-intersecting*** **family of subsets of [n] with**

$|F| =$

$C(n, s) + C(n, s-1) + \ldots + C(n, 0).$

Snevily Theorem (2003):

Let $L = \{ l_1, l_2, \ldots , l_s \}$ be a set of s positive integers. If F is a *L-intersecting* family of subsets of [n], then

$$|F| \leqslant C(n-1, s) + C(n-1, s-1) + \ldots + C(n-1, 0).$$

Example:

Let $L = \{ 1, 2, \ldots , s \}$ and

F be the set of subsets of [n] of sizes $\leqslant s + 1$ which contain 1.

Then F is a *L-intersecting* family of subsets of [n] with

$$|F| = C(n-1, s) + C(n-1, s-1) + \ldots + C(n-1, 0).$$

Question:

Let $L = \{ l_1, l_2, \ldots , l_s \}$ be a set of s positive integers. If F is a *L-intersecting* family of k-subsets of [n], then

$$|F| \leqslant C(n-1, s) \ ?$$

Alon-Babai-Suzuki Theorem:

Let $L = \{ l_1, l_2, \ldots, l_s \}$ be a set of s nonnegative integers and $K = \{k_1, k_2, \ldots, k_r \}$ with each $k_i > s - r$. If F is a *L-intersecting* family of subsets of [n] such that $|F_i| \in K$ for each $F_i \in F$, then

$|F| \leqslant C(n, s) + C(n, s-1) + \ldots + C(n, s-r+1)$.

- **Example:**

Let *F* be the set of subsets of [n] of sizes at least s – r +1 and at most s

$$L = \{ 0, 1, \dots, s-1 \}$$

Then F **is a *L-intersecting* family of subsets of [n] with**

$$|F| = C(n, s) + C(n, s-1) + \dots + C(n, s-r+1).$$

Snevily's Conjecture A:

Let p be prime , $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ *be two disjoint subsets of* $\{0, 1, 2, \ldots, p-1\}$.

Suppose $F = \{F_1, F_2, \ldots, F_m\}$ *is a family of subsets of* $[n]$ *such that*

$|F_i \cap F_j| \pmod{p} \in L$ *for* $i \neq j$

$|F_i| \pmod{p} \in K$ *for every* i.

Then $|F| \leqslant C(n, s)$.

Theorem (Chen and Liu) :

Let p *be prime*, $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ *be two disjoint subsets of* $\{0, 1, 2, \ldots, p-1\}$.

Suppose $F = \{F_1, F_2, \ldots, F_m\}$ *is a family of subsets of* $[n]$ *such that*

$|F_i \cap F_j| \pmod{p} \in L$ *for* $i \neq j$

$|F_i| \pmod{p} \in K$ *for every* i.

Then $|F| \leqslant$

$C(n-1, s) + \ldots + C(n-1, s-2r+1)$

Conjecture A is true when $r = 1$:

Theorem. Let $L = \{l_1, l_2, \ldots, l_s\}$ *be a subset of* $\{0, 1, 2, \ldots, p-1\} \setminus \{k\}$.

Suppose $F = \{F_1, F_2, \ldots, F_m\}$ *is a family of k-subsets of* $[n]$ *such that* $|F_i \cap F_j| \pmod{p} \in L$ *for* $i \neq j$

Then $|F| \leqslant C(n, s)$

$$= C(n-1, s) + C(n-1, s-1).$$

Snevily's Conjecture B:

Let $L = \{ l_1, l_2, \ldots, l_s \}$ *be a set of s positive integers. Suppose*
$A = \{ A_1, A_2, \ldots, A_m \}$ *and*
$B = \{ B_1, B_2, \ldots, B_m \}$ *are two families of subsets of [n] such that*

$|A_i \cap B_j| \in L$ *for* $i \neq j$

$|A_i \cap B_i| = 0$ *for every* i.

Then $m \leqslant C(n, s)$.

Theorem (Chen and Liu) :

$A = \{ A_1, A_2, \ldots , A_m \}$

$B = \{ B_1, B_2, \ldots , B_m \}$

satisfy conditions in Conjecture B

If either A is k-uniform or B is k-uniform, then Conjecture B is true.

Conjecture C (Liu &Liu):

Let $L = \{ l_1, l_2, \ldots, l_s \}$ *be a set of s positive integers. Suppose*
$A = \{ A_1, A_2, \ldots, A_m \}$ *and*
$B = \{ B_1, B_2, \ldots, B_m \}$ *are two families of subsets of [n] such that*
$|A_i \cap B_j| \in L$ *for* $i \neq j$

Then $m \leqslant$

$$C(n - 1, s) + \ldots + C(n - 1, 0).$$

When $A_i = B_i$, Conjecture C gives Snevily Theorem:

Let $L = \{ l_1, l_2, \ldots , l_s \}$ be a set of s positive integers. If F is a *L-intersecting* family of subsets of [n], then

$$|F| \leqslant C(n-1, s) + C(n-1, s-1) + \ldots + C(n-1, 0).$$

- **定理 (Frankl and Wilson, 1981）**

设 $L = \{l_1, l_2, \ldots\ldots, l_s\}$. 如果 F 是一个 [n] 上的 L-相交的子集族, 那么

$$|F| \leqq C(n, s) + C(n, s-1) + \ldots + C(n, 0) .$$

- 证明：设 $F = \{F_1, F_2, \ldots\ldots, F_m\}$ with

$$|F_1| \leqq |F_2| \leqq \ldots\ldots \leqq |F_m| .$$

设 v_i 为 F_i 的特征涵数:

$$\begin{cases} v_{i,j} = 1 & \text{如果 } j \in F_i \\ v_{i,j} = 0 & \text{否则} \end{cases}$$

- $v_i \cdot v_j = |F_i \cap F_j|$

- 定义多元线性函数如下：

$$f_i(x) = \prod_{l_k<|F_i|}(v_i \cdot x - l_k)$$

$$f_i(v_i) \neq 0,\quad i = 1, 2, \ldots, m$$

$$f_i(v_j) = 0 \quad \text{如果} \quad i > j$$

- 我们断定：

$f_1(x), f_2(x), \ldots, f_m(x)$ 是线性无关的

设 $c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x) = 0$

假定 i_0 是最小的下标使得

$$c_{i_0} \neq 0$$

代入 v_{i_0} 得到 $c_{i_0} f_{i_0}(v_{i_0}) = 0$

因为 $f_{i_0}(v_{i_0}) \neq 0$，所以 $c_{i_0} = 0$ 茅盾

- 每一个 $f_i(x)$ 是下列之项的线性组合：

$$x_{i_1}x_{i_2}x_{i_3}\ldots x_{i_t}\,, \qquad t \leqq s$$

总共有

$$C(n, s) + C(n, s-1) + \ldots + C(n, 0)$$

个线性无关的那些项，所以

$$|F| = m \leqq C(n, s) + C(n, s-1) + \ldots + C(n, 0)\,.$$

- 定理 (Ray-Chaudhuri and Wilson）

设 $L = \{l_1, l_2, \ldots, l_s\}$. 如果 F 是一个 [n] 上的 L-相交的 k-子集族, 那么

$$|F| \leq C(n, s) .$$

- 证明：设 $F = \{F_1, F_2, \ldots\ldots, F_m\}$

设 v_i 为 F_i 的特征涵数

$$v_i \cdot v_j = |F_i \cap F_j|$$

定义多元线性函数如下：

$$f_i(x) = \prod_{l_k<|F_i|}(v_i \cdot x - l_k)$$

$$f_i(v_i) \neq 0\,,\quad i = 1, 2, \ldots, m$$

$$f_i(v_j) = 0 \quad \text{如果} \quad i \neq j$$

- 设Q为所有大小不超过 s − 1 的子集的集合: $Q = \{ I \mid |I| \leqslant s - 1 \}$

  $$|Q| = C(n, s-1) + \ldots + C(n, 0)$$

- 对每一个 $I \in Q$，定义：

$$g_I(x) = \left(\sum_{i=1}^{n} x_i - k \right) \prod_{j \in I} x_j$$

$g_I(x) : I \in Q$

$f_1(x), f_2(x), \ldots, f_m(x)$

是线性无关的

$$|F| + |Q|$$

$$\leqq C(n, s) + C(n, s-1) + \ldots + C(n, 0)$$

$$|F| \leqq C(n, s) + C(n, s-1) + \ldots + C(n, 0)$$
$$- (C(n, s-1) + \ldots + C(n, 0))$$
$$= C(n, s)$$